Existence of Bubbly Equilibria in Overlapping Generations Models with Stochastic Production*

Marten Hillebrand
Karlsruhe Institute of Technology (KIT)
Department of Economics and Business Engineering
Kollegium IV am Schloß
D-76128 Karlsruhe, Germany
marten.hillebrand@kit.edu
Phone: +49 (0) 721-608 45667
Fax: +49 (0) 721-608 43082

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Abstract

The paper develops a dynamical systems approach to study asset bubbles in OLG economies with stochastic production. We derive necessary and sufficient conditions for bubbly equilibria to exist and characterize the maximum sustainable bubble. Even if they exist, bubbles are temporary and the economy converges to a bubbleless equilibrium with probability one. We also demonstrate that the existence conditions can be relaxed if frictions such as borrowing constraints are introduced.

Keywords: Asset bubbles, OLG, stochastic production, capital accumulation, dynamical systems, borrowing constraints.

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Introduction

When do asset prices exceed the fundamental value of the underlying asset? This phenomenon of a so-called asset bubble has long been studied in the literature. Numerous papers provide conditions under which bubbles are compatible with rational, fully informed investors and study their consequences for the real economy. A common feature of almost all these studies, however, is that they employ a deterministic framework. The main contribution of the present paper is to study asset bubbles in stochastic economies where the production side is subjected to random productivity shocks. Using methods from dynamical systems theory, we derive conditions under which asset bubbles exist in a broad class of such economies. As the formal approach to be developed seems applicable also in other and more general situations, the paper also offers a methodological contribution.

A natural framework to study asset bubbles and their equilibrium implications is the class of overlapping generations (OLG) models on which the present paper will exclusively focus. A first class of models in this literature studies monetary bubbles corresponding to valued fiat money in models of pure exchange. Early studies of deterministic exchange economies may be found in Gale (1973), Okuno & Zilcha (1983), or Beneviste & Cass (1986). These papers show that monetary bubbles can only exist if the non-monetary equilibrium is non-optimal. The results were generalized, e.g., in Koda (1984), Manuelli (1990), or Aiyagari & Peled (1991) to stochastic exchange economies where incomes follow exogenous random processes and consumers may have access to an intertemporal storage technology. The analysis to be presented in this paper will show that the existence conditions in Manuelli (1990) are structurally similar to the ones for a stochastic production economy.
Conceptually, most of the previous and related approaches focus on stationary equilibria for which they offer abstract existence results. Issues such as dynamic stability and the role played by initial conditions are typically not studied. A notable exception is Rochon & Polemarchakis (2006) who extend the deterministic OLG model with pure exchange to include a financial sector that issues money in exchange for debt and conduct a full-fledged analysis of the resulting dynamics. The present paper attempts to conduct a study in the same spirit for a stochastic economic environment.

A second class of models includes an explicit description of the production process and the accumulation of capital. This permits to study the impact of asset bubbles on production and investment in the economy. For these economies, Tirole (1985) showed that asset bubbles occur if and only if the bubbleless equilibrium is inefficient due to an overaccumulation of capital. In situations where the bubbleless equilibrium does not suffer from over-accumulation, bubbles may still exist in the presence of frictions. Michel & Wigniolle (2003) study a monetary OLG model with production where consumers hold money due to cash-in advance constraints. They show that temporary bubbles may exist even if the moneyless equilibrium fails to exhibit overaccumulation of capital. Similarly, Kunieda (2008) shows that asset bubbles can emerge in economies with overaccumulation where consumers face borrowing constraints. Below we will discuss how the deterministic results in Kunieda (2008) extend to the stochastic setting of this paper.

An issue closely related to the emergence of a bubble is the sustainability of governmental debt which may be viewed as a bubble rolled over from generation to generation. The differences between debt and bubbles are thoroughly exhibited in de la Croix & Michel (2002, p.212). Starting with the seminal paper by Diamond (1965), several papers focus on the sustainability and optimality of government debt, see de la Croix & Michel (2002) for a survey. Typically, however, theses studies are also placed in a deterministic setting. An exception may be found in Bertocchi (1994), who studies a stochastic OLG economy with government debt offering a safe return. Her model constitutes a special case of the framework to be developed in this paper and we will comment on her findings below.

To account for aggregate fluctuations of the type observed over the business cycle, most macroeconomic models incorporate random shocks, in particular productivity shocks. For OLG production economies, such a setup was introduced in Wang (1993) and further generalized, e.g., in Wang (1994), Morand & Reffett (2007), McGovern et al. (2013), or Hillebrand (2014). Extending the previous studies of bubbles to such a random environment seems important not only to incorporate business cycle fluctuations, but also because the results for deterministic economies indicate that bubbles are relatively fragile and their emergence is subject to initial conditions. Thus, it seems important to analyze whether the deterministic findings are robust and continue to hold in a random setting.

To the author's best knowledge, a general study of bubbles in OLG economies with random production and endogenous capital accumulation is still missing in the literature. Filling this gap is therefore the primary contribution of this paper. While the fundamental side of the economy will be similar to Wang (1993), we will argue below how and why the results and methods should also carry over to more general classes of economies. Conceptually, the paper develops and applies a dynamical systems approach suitably adapted to a random environment. This preserves the main strength of Tirole (1985) whose existence conditions are essentially based on the dynamic properties of the equilibrium mapping. In particular, the saddle-path towards the bubbly steady state defines the maximum sustainable bubble under which the state dynamics remain bounded in Tirole's model. In the stochastic case studied here, matters are considerably more complicated as the equilibrium bubble must be sustainable under *any* sequence of shocks. For this reason, the existence conditions derived in this paper are based on the dynamic properties of an *entire family* of equilibrium mappings parameterized in the shock. This structure provides a natural extension of the deterministic dynamical system in Tirole (1985) to the present stochastic setting. As a consequence, the existence conditions derived below become natural and intuitive generalizations of the ones in Tirole (1985) which can be recovered as a special case.

From a purely methodological standpoint, the paper analyzes equilibria which are generated by randomly mixing a family of mappings each of which possesses an interior fixed point which is saddle-path stable. This is a situation that arises in many macroeconomic models (for example, in the stochastic neoclassical growth model in state-space form) and the approach to be developed delivers simple and geometrically intuitive conditions under which such a system generates bounded dynamics and possesses stable, self-supporting sets. Using the stable manifold theorem (cf. Nitecki (1971)), the key ingredient is a complete characterization of the regions in the state space in which each mapping generates stationary dynamic behavior. Thus, great care is placed on a clean mathematical characterization of these regions (cf. Lemma 3.4 in Section 3). The methods to be employed seem applicable also in other and more general situations and could, therefore, be of some general methodological interest quite independent of the particular theme of this paper.

The analysis of this paper unfolds as follows. In a first step, we impose restrictions under which bubbly equilibria are generated by randomly mixing a family of dynamic mappings on a suitably defined state space. This structure provides the basis for applying dynamical systems theory to study bubbly equilibria. In a second step, we characterize the dynamic properties of each member of this family and whether it displays expansive or stationary behavior. This permits to completely characterize the model's dynamic behavior under arbitrary sequences of shocks and for different initial conditions. In particular, it will allow us to derive necessary and sufficient conditions for bubbly equilibria to exist and derive an upper bound on the maximum initial bubble that can be sustained over time under any sequence of shocks. Essentially, our existence conditions require the state dynamics to be exclusively generated by stationary dynamic mappings each of which generates bounded dynamics on a certain subset of the state space. The intersection of these ranges defines an upper bound for the maximum initial bubble that can be sustained over time just as in Tirole (1985). We also show that even if they exist, bubbles are temporary in the sense that generically the economy converges to a bubbleless situation with probability one. Finally, we demonstrate that our existence conditions can be relaxed if frictions such as borrowing constraints are introduced.

The paper is organized as follows. Section 1 introduces the model. Section 2 derives the structure of equilibria which are generated by a family of mappings whose dynamic properties are analyzed in Section 3. Section 4 establishes necessary and sufficient conditions for bubbly equilibria to exist and discusses various extensions of the model. Section 5 modifies the previous setup to study the role of borrowing constraints. Section 6 concludes. All proofs are placed in the Mathematical Appendix.

1 The Model

Production sector.
The production side consists of a representative firm which operates a linear-homogeneous technology to produce an all-purpose consumption good using labor and capital as inputs. In addition, the production process is subjected to an exogenous TFP-shock ε_t in each period $t \geq 0$. At equilibrium, labor supply will be constant and normalized to unity such that per-capita output y_t is determined from capital k_t and the current shock according to the intensive form technology $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$

$$y_t = \varepsilon_t f(k_t). \tag{1}$$

The function f is C^2 with $f(0) = 0$ and derivatives satisfying $f'' < 0 < f'$ and the Inada conditions $\lim_{k \searrow 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$. The shock process $\{\varepsilon_t\}_{t \geq 0}$ consists of independent random variables where each ε_t is distributed according to the probability measure ν supported on the compact set $\mathcal{E} \subset \mathbb{R}_{++}$. This structure induces a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables are defined and a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ to which all equilibrium processes considered below are adapted.[1] Denote by $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ the expectations operator conditional on the information represented by $\mathcal{F}_t$ and $\mathbb{E}_\nu[\cdot]$ the expectation with respect to ν.[2]

[1] Formally, a stochastic process $\{\xi_t\}_{t \geq 0}$ taking values in some set $\Xi \subset \mathbb{R}^M$ is adapted to the filtration defined if each random variable $\xi_t : \Omega \longrightarrow \Xi$ is Borel-measurable with respect to $\mathcal{F}_t$ and hence depends only on shocks up to time t.

[2] In the following analysis, all equalities or inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely without further notice. Measurability of mappings always refers to the Borel σ algebras.

Under profit maximization and perfect competition on factor markets, the equilibrium wage w_t and capital return r_t are determined by the standard formulas

$$w_t = \mathcal{W}(k_t;\varepsilon_t) := \varepsilon_t\,[f(k_t) - k_t f'(k_t)] \tag{2a}$$
$$r_t = \mathcal{R}(k_t;\varepsilon_t) := \varepsilon_t f'(k_t). \tag{2b}$$

Consumption sector.
The consumption sector consists of overlapping generations of homogeneous consumers who live for two periods. Abstracting from population growth, the size of each generation can be normalized to one. A young consumer in period t is endowed with one unit of labor time which is supplied inelastically to the labor market. Old consumers own the existing stock of capital which they supply to the production process.
A young consumer in period $t \geq 0$ earns labor income $w_t > 0$ part of which is consumed and the remainder invested. For the latter purpose, the consumer can invest in capital which yields the random capital return r_{t+1}. In addition, a bubbly asset is available which promises the random return $r^\star_{t+1}$ to be paid in $t+1$ per unit invested at time t.
Let s_t and b_t be the investments in capital and the bubble at time $t \geq 0$. These choices define first period consumption $c^1_t = w_t - b_t - s_t$ while second period consumption is given by the random variable $c^2_{t+1} = b_t\, r^\star_{t+1} + s_t\, r_{t+1}$. Here the randomness enters through the uncertain returns on both investments which are treated as given random variables in the decision. As in Wang (1993), young consumers evaluate the expected utility of different consumption plans (c^1_t, c^2_{t+1}) using an additive von-Neumann Morgenstern utility $U(c^y, c^o) = u(c^y) + v(c^o)$. Each $z \in \{u, v\}$ is C^2 with derivatives satisfying $z'' < 0 < z'$ and the one-sided Inada condition $\lim_{c\searrow 0} z'(c) = \infty$.
Each young consumer chooses investment to maximize her expected lifetime utility. The decision problem reads:

$$\max_{b,s}\Big\{ u(w_t - b - s) + \mathbb{E}_t\left[v\big(r^\star_{t+1}\, b + r_{t+1}\, s\big)\right] \,\Big|\, s \geq 0, b + s \leq w_t \Big\}. \tag{3}$$

Note that no short-selling constraints on b are imposed at the individual level. Thus, any solution to (3) satisfies the corresponding first order conditions.
At equilibrium, the investment in capital s_t determines next period's capital stock

$$k_{t+1} = s_t. \tag{4}$$

Denote by $b_t \geq 0$ the value of the bubble in period $t \geq 0$. No resources are added or withdrawn from outside such that the bubble must be completely self-financing, i.e.

$$b_{t+1} = r^\star_{t+1}\, b_t, \quad t \geq 0. \tag{5}$$

Old consumers in period $t \geq 0$ simply consume the proceeds of their investments in bubbles and capital made during the previous period.

Equilibrium.
The economy is $\mathscr{E} = (f, \nu, u, v)$ plus initial conditions. The following definition of a bubbly equilibrium reconciles market clearing, individual optimality, and rational expectations. Note that the Inada conditions imposed above ensure an interior equilibrium allocation of capital and consumption of both generations.

Definition 1

Given $b_0 \geq 0$, $k_0 > 0$, and $\varepsilon_0 \in \mathcal{E}$, an equilibrium of $\mathscr{E}$ is an adapted stochastic process $\{w_t, r_t, r_t^\star, b_t, s_t, k_{t+1}\}_{t\geq 0}$ of non-negative values which satisfies the following for each $t \geq 0$:

(i) The pair (b_t, s_t) solves (3) at the given wage and returns while k_{t+1} follows from (4).

(ii) Factor prices w_t and r_t are determined by (2a,b) and b_t evolves according to (5).

The equilibrium is called bubbly, if $b_t > 0$ and bubbleless if $b_t = 0$ for all $t \geq 0$.

Additional restrictions.
The subsequent analysis will frequently impose additional restrictions on the economy $\mathscr{E}$. As these conditions are somewhat stronger than the ones imposed above, it will explicitly be indicated when they are used.
Denote by $E_h(x) := |xh'(x)/h(x)|$, $x \in \mathbb{D} \subset \mathbb{R}$ the (absolute) elasticity of a differentiable function $h : \mathbb{D} \longrightarrow \mathbb{R}\backslash\{0\}$. Additional restrictions on the utility functions u and v are:

$$(U1)\ E_{v'} \leq 1 \quad (U2)\ \lim_{c\to\infty} c\, v'(c) = \infty \quad (U3)\ E_{v'} \equiv \theta \quad (U4)\ E_{u'} \leq 1 \quad (U5)\ \lim_{c\to\infty} u'(c) = 0.$$

Examples satisfying (U1) and (U2) are power utility $v(c) = \theta^{-1}c^\theta$, $0 < \theta < 1$, or CES utility $v(c) = [1 - \theta + \theta c^\beta]^{1/\beta}$, $0 < \theta < 1$, $\beta > 0$. While (U2) excludes logarithmic utility, the first example shows that this case can still be approximated by letting $\theta \to 0$. Under (U3), second period utility v exhibits constant relative risk aversion while (U4) is automatically satisfied if (U1) holds and $v(c) = \beta u(c)$, $\beta > 0$. The restriction (U5) on the boundary behavior of u' is standard.
Additional restrictions imposed on the production technology f are the following:

$$(T1)\ E_{f'} \leq 1 \qquad\qquad (T2)\ E_f < \frac{1}{2}.$$

Restriction (T1) is known as *capital income monotonicity* and widely used in OLG models with production, cf. Wang (1993), de la Croix & Michel (2002), or Hauenschild (2002). It holds, e.g., for a Cobb-Douglas technology $f(k) = k^\alpha$, $0 < \alpha < 1$. The second restriction (T2) ensures that labor income throughout exceeds capital income, which is a well-established empirical regularity. In the Cobb-Douglas case, it holds if $\alpha < \frac{1}{2}$.

2 Equilibrium Dynamics

Risk structure of bubbles.
While the general definition of a bubbly equilibrium from the previous section imposes no restrictions on the risk structure of the return process $\{r_t^\star\}_{t\geq 0}$, the following analysis assumes that the bubble return offered at time t is of the following form

$$r_{t+1}^\star = \mathcal{R}^\star(z_t; \varepsilon_{t+1}) := \vartheta(\varepsilon_{t+1})\, z_t, \quad t \geq 0. \tag{6}$$

Here $z_t > 0$ is determined in period t and $\vartheta : \mathcal{E} \longrightarrow \mathbb{R}_{++}$ is a bounded measurable function which defines the *risk-structure* of the bubbly asset. Two specific cases are of particular interest. If $\vartheta \equiv \bar{\vartheta}$ the bubble offers a riskless return. If $\vartheta = \mathrm{id}_{\mathcal{E}}$, the identity map on

$\mathcal{E}$, the returns on bubbles exhibit the same risk structure as capital investments. This will be referred to as a *capital-equivalent bubble*. In the latter case, one necessarily has $z_t = f'(k_{t+1})$ which implies $r^{\star}_{t+1} \equiv r_{t+1}$ for each $t \geq 0$, i.e., the returns on bubbles and capital coincide pointwise.

A straightforward interpretation of (6) is as follows. Suppose there are finitely many shocks $\mathcal{E} = \{\varepsilon^1, \ldots, \varepsilon^M\}$ and in each period there exists a complete set of M Arrow securities. Let $p^m_t > 0$ be the price of security m that pays off one unit in $t+1$ iff $\varepsilon_{t+1} = \varepsilon^m$. In each period $t \geq 0$, the institution backing the bubbly asset (e.g., some government or an investment fund) issues a portfolio $a_t = (a^m_t)_{m=1,\ldots,M} \in \mathbb{R}^M_{++}$ of these securities to finance the bubble, i.e., $\sum_{m=1}^{M} a^m_t p^m_t = b_t$. Let the mix of securities be constant over time and determined by ϑ where $\vartheta^m := \vartheta(\varepsilon^m)$ is the relative share of security m in the portfolio. The scalar z_t then determines the supply of security m as $a^m_t = b_t z_t \vartheta^m$. For young consumers to be willing to buy these assets, prices must satisfy the Euler equations $p^m_t = \nu(\{\varepsilon^m\}) v'(a^m_t + \varepsilon^m f'(k_{t+1})k_{t+1})/u'(w_t - b_t - k_{t+1})$. Combined with the first order conditions for an expectations-consistent capital investment derived from (3) this yields precisely the Euler equation (8) derived below. All these arguments also extend to an infinite set $\mathcal{E}$ and a continuum of Arrow securities. Extensions of (6) towards more general bubble returns with state-dependent risk structure are discussed in Section 4.

Recursive equilibrium structure.
To uncover the recursive structure of equilibria, consider an arbitrary period $t \geq 0$. Let the current state $x_t := (w_t, b_t)$ determined by (2a) and (5) be given and $w_t > b_t \geq 0$. The temporary equilibrium problem for period t is to determine next period's capital $k_{t+1} > 0$ and a value $z_t > 0$ consistent with an optimal savings decision derived from (3) and rational, self-confirming expectations. The scalar z_t determines the ex-ante bubble return $r^{\star}_{t+1}$ offered at time t according to (6) such that young consumers are willing to absorb the current bubble. Combining the first order conditions[3] of (3) with (2b), (4), and (6), define for $i \in \{1, 2\}$ the mappings $H^{(i)}(\cdot, \cdot; w, b) : \mathbb{R}_{++} \times]0, w - b[\longrightarrow \mathbb{R}$

$$H^{(1)}(z, k; w, b) := u'(w - b - k) - \mathbb{E}_\nu\left[\mathcal{R}(k; \cdot) v'\big(b\,\mathcal{R}^{\star}(z; \cdot) + k\mathcal{R}(k; \cdot)\big)\right] \quad (7a)$$

$$H^{(2)}(z, k; w, b) := u'(w - b - k) - \mathbb{E}_\nu\left[\mathcal{R}^{\star}(z; \cdot) v'\big(b\,\mathcal{R}^{\star}(z; \cdot) + k\mathcal{R}(k; \cdot)\big)\right]. \quad (7b)$$

Then, given $w_t > b_t \geq 0$ the previous problem reduces to solving the Euler equations

$$H^{(1)}(z_t, k_{t+1}; w_t, b_t) = H^{(2)}(z_t, k_{t+1}; w_t, b_t) = 0. \quad (8)$$

The following result establishes conditions under which a unique solution to (8) exists.

Lemma 2.1
Let the additional restrictions (T1), (U1), and (U2) hold. Then, for each $w > b \geq 0$ there exist unique values $z > 0$ and $0 < k < w - b$ such that $H^{(1)}(z, k; w, b) = H^{(2)}(z, k; w, b) = 0$.

Properties (U1) and (T1) ensure that an increase in the returns r_{t+1} or $r^{\star}_{t+1}$ offered at time t increases the desired investment in capital respectively bubbles. Economically, this means that the intertemporal substitution effect always dominates the income effect. These conditions appear to be minimal ingredients under which the state dynamics derived

[3] Throughout this paper, we exploit that differentiation may be interchanged with the expectations operator $\mathbb{E}_\nu[\cdot]$ if the integrand is continuously differentiable and integration is over a compact set.

below are well-defined, i.e., each state has a unique successor. If $b_t = 0$, either of the two restrictions alone is sufficient. The additional restriction (U2) ensures that consumers are willing to absorb any bubble b_t not exceeding their income w_t if they are offered a sufficiently large return. This permits to define the model's state space as in (9) below which is the 'largest' state space possible. If (U2) failed to hold – as in the example with log-utility in Section 4 – tighter bounds on the bubble would be needed.[4]

Unless stated otherwise, the remainder assumes that the hypotheses of Lemma 2.1 hold. This permits to define the model's *endogenous state space* as

$$\mathbb{X} := \Big\{(w, b) \in \mathbb{R}^2_+ \,|\, w > b\Big\}. \tag{9}$$

Exploiting the result from Lemma 2.1, let the mappings $\mathcal{K} : \mathbb{X} \longrightarrow \mathbb{R}_{++}$ and $\mathcal{Z} : \mathbb{X} \longrightarrow \mathbb{R}_{++}$ determine the solutions k_{t+1} and z_t to (8) for each $x_t = (w_t, b_t) \in \mathbb{X}$. Using the implicit function theorem, the following result shows that these mappings are smooth (continuously differentiable) and characterizes their monotonicity and boundary behavior. These properties provide the basis for the dynamical systems approach developed below, which will make repeated use of the Grobman-Hartman Theorem and the Stable Manifold Theorem in order to characterize the dynamic behavior of the equilibrium mappings.

Lemma 2.2

If (T1), (U1), and (U2) hold, both $\mathcal{K}$ and $\mathcal{Z}$ are C^1 and satisfy the following properties:

(i) $\lim_{w-b\searrow 0} \mathcal{K}(w, b) = 0$ *and* $\lim_{w-b\searrow 0} \mathcal{Z}(w, b) = \infty$. *(ii)* $0 < \mathcal{K}_w < -\mathcal{K}_b$.

(iii) If, in addition, either $\vartheta = \mathrm{id}_{\mathcal{E}}$ or (U3) holds, then $0 < -\mathcal{Z}_w < \mathcal{Z}_b$ and $\mathcal{K}_w\mathcal{Z}_b \geq \mathcal{K}_b\mathcal{Z}_w$.

Lemma 2.2 (ii) shows that capital investment increases with income and decreases with the size of the bubble. The latter is the standard crowding-out effect which is well-known from deterministic models. Similarly, (iii) shows that the return required for consumers to be willing to absorb the current bubble increases with its size and decreases with income. The main ingredient to the proof of (iii) is Lemma B.1 which requires second-period utility to display constant relative risk aversion. While this is a rather strong restriction, numerical experiments with utility functions v not satisfying (U3) have throughout displayed the same properties of $\mathcal{Z}$ as in Lemma 2.2 (iii) suggesting that this restriction could probably be relaxed. If the bubble is capital-equivalent, no such condition is needed.

Equilibrium dynamics.

Combining Lemma 2.1 with (2a), (5), and (6) the evolution of the endogenous state variable under the exogenous shocks is governed by the map $\Phi = (\Phi^{(1)}, \Phi^{(2)}) : \mathbb{X} \times \mathcal{E} \longrightarrow \mathbb{R}^2_+$,

$$w_{t+1} = \Phi^{(1)}(w_t, b_t; \varepsilon_{t+1}) := \mathcal{W}(\mathcal{K}(w_t, b_t), \varepsilon_{t+1}) \tag{10a}$$
$$b_{t+1} = \Phi^{(2)}(w_t, b_t; \varepsilon_{t+1}) := \mathcal{R}^\star(\mathcal{Z}(w_t, b_t); \varepsilon_{t+1})b_t. \tag{10b}$$

As $\Phi(\cdot; \varepsilon)$ does not map $\mathbb{X}$ into itself, we refer to it as a *pseudo-dynamical system*. This feature is essentially due to the boundary behavior stated in Lemma 2.2 (i) and is well-known from deterministic models with bubbles, cf. Tirole (1985). Given an initial state

[4] In Tirole (1985) or Weil (1987), restrictions are imposed on derived objects such as the savings function or the factor pricing functions $\mathcal{W}$ and $\mathcal{R}$ and it seems not clear how they restrict the underlying class of preferences and technology. For instance, Weil (1987) assumes that the interest elasticity of savings is positive, which is exactly what is ensured by (U1) and (T1).

$x_0 = (w_0, b_0) \in \mathbb{X}$, any equilibrium process $\{x_t\}_{t\geq 0}$ is generated by randomly mixing the family of mappings $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ defined in (10a,b). That is, for each $t > 0$ the realization of the production shock ε_t 'selects' a particular map that determines the state x_t from its previous value x_{t-1}. Structurally, this corresponds to a two-dimensional version of the one-dimensional dynamics in Wang (1993). The endogenous state variables $\{x_t\}_{t\geq 0}$ together with the exogenous shock process $\{\varepsilon_t\}_{t\geq 0}$ completely determine the other equilibrium variables of the model. Therefore, the existence of equilibrium is equivalent to determining $x_0 \in \mathbb{X}$ such that the process generated by (10a,b) satisfies $x_t \in \mathbb{X}$ for all $t \geq 0$ under $\mathbb{P}$-almost all paths of the noise process. Since $b_0 = 0$ implies $b_t = 0$ for all $t > 0$, the economy has a unique bubbleless equilibrium along which the state dynamics reduce to a one-dimensional system given by $w_{t+1} = \mathcal{W}(\mathcal{K}(w_t, 0), \varepsilon_{t+1})$, $t \geq 0$. This is precisely the equilibrium studied in Wang (1993). It will turn out in the following sections that the properties of the bubbleless equilibrium are crucial for the existence of bubbly equilibria, a finding in line with the results obtained in Tirole (1985) for a deterministic economy.

3 Stationary and Expansive Mappings

Structure of dynamic mappings.
From the structure derived in the previous section, it stands to reason that the existence of bubbly equilibria depends crucially on the dynamic properties of the mappings $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ defined in (10a,b). In this section, we fix a value $\varepsilon \in \mathcal{E}$ to study the dynamic properties of the single map $\Phi := \Phi(\cdot;\varepsilon)$. Mathematically, this corresponds to analyzing the model's behavior under a particular realization of shocks given by the constant sequence $(\varepsilon, \varepsilon, \varepsilon, \ldots)$.[5] Define the state space $\mathbb{X}$ as in (9) and consider the pseudo-dynamical system $\Phi : \mathbb{X} \longrightarrow \mathbb{R}^2_+$,

$$\Phi(w, b) = \begin{pmatrix} \phi(w, b) \\ \psi(w, b)b \end{pmatrix}. \tag{11}$$

Throughout, the following restrictions will be imposed on ϕ and ψ .

Assumption 1
The maps $\phi : \mathbb{X} \longrightarrow \mathbb{R}_{++}$ and $\psi : \mathbb{X} \longrightarrow \mathbb{R}_{++}$ in (11) are C^1 with derivatives $0 < \phi_w < -\phi_b$, $0 < -\psi_w < \psi_b$ and $\phi_w\psi_b \geq \phi_b\psi_w$. Also, $\lim_{w-b\searrow 0} \phi(w, b) = 0$ and $\lim_{w-b\searrow 0} \psi(w, b) = \infty$.

For $t \geq 0$, define the t-fold composition Φ^t recursively by setting $\Phi^0 := \mathrm{id}_{\mathbb{X}}$ and $\Phi^t(x) := \Phi \circ \Phi^{t-1}(x)$ for all $x \in \mathbb{X}$ where it is defined. Let $\mathbb{X}_+ := \mathbb{X} \bigcap \mathbb{R}^2_{++}$ and $\mathbb{X}_0 := \mathbb{X}\backslash\mathbb{X}_+$. The second equation in (11) reveals that $\mathbb{X}_0$ is self-supporting under Φ, i.e., $\Phi(\mathbb{X}_0) \subset \mathbb{X}_0$. The following assumption restricts the dynamic behavior of Φ on $\mathbb{X}_0$ which will further be discussed in the next section.

Assumption 2
Φ has a unique fixed point $\bar{x}^0$ in $\mathbb{X}_0$. This fixed point satisfies $\phi_w(\bar{x}^0) < 1$.

As the dynamics on $\mathbb{X}_0$ are one-dimensional, uniqueness of the fixed point and the second condition in Assumption 2 ensure that $\lim_{t\to\infty} \Phi^t(x) = \bar{x}^0$ for all $x \in \mathbb{X}_0$, i.e., $\bar{x}^0$ is globally asymptotically stable on $\mathbb{X}_0$.

[5] Note that this does not say that the distribution ν of the shocks is degenerate, i.e., consumers continue to maximize expected utility such that this case is *not* the one studied in Tirole (1985).

Stationarity.
Our goal will be to characterize the qualitative dynamic behavior of Φ on $\mathbb{X}_+$. Specifically, we want to distinguish cases where Φ generates expansive respectively stationary behavior. This distinction is based on the following

Definition 2
Φ is called stationary, if it has a fixed point in $\mathbb{X}_+$. Otherwise, it is called expansive.

The idea of stationarity of a map is that there is at least one state $x \in \mathbb{X}_+$ which is sustainable in the sense that $\Phi^t(x) \in \mathbb{X}_+$ for all $t \geq 0$. The merit of Assumption 2 is that it permits the following characterization of stationarity.

Lemma 3.1
Under Assumptions 1 and 2, a map Φ of the form (11) is stationary, if and only if $\psi(\bar{x}^0) < 1$.

Excluding the non-generic case $\psi(\bar{x}^0) = 1$, the next result shows that a sustainable state fails to exist if Φ is expansive, i.e., the dynamics will leave the state space $\mathbb{X}$ in finite time for any initial value $x_0 \in \mathbb{X}_+$. In this sense, any initial value which has $b_0 > 0$ is unsustainable under an expansive mapping Φ.

Lemma 3.2
Let Assumptions 1 and 2 hold and assume that the fixed point $\bar{x}^0 \in \mathbb{X}_0$ satisfies $\psi(\bar{x}^0) \neq 1$. If Φ is expansive, then for each $x_0 \in \mathbb{X}_+$ there exists $t_0 \in \mathbb{N}$ such that $\Phi^{t_0}(x_0) \notin \mathbb{X}$.

From the restrictions imposed so far, it does not seem possible to infer that a stationary map Φ has a unique steady state in $\mathbb{X}_+$. However, it will turn out that such a uniqueness property is valuable if not required to further describe the qualitative behavior of stationary mappings. For this reason, we impose uniqueness directly by the following assumption. In addition, we rule out non-hyperbolic steady states by assuming that no Eigenvalue λ of the Jacobian matrix $D\Phi(\bar{x})$ satisfies $|\lambda| = 1$. Conditions under which these restrictions are consistent with the primitives of the model are discussed in the next section.

Assumption 3
Φ has at most one steady state in $\mathbb{X}_+$. Moreover, if it exists, this steady state is hyperbolic.

A first step towards characterizing the global dynamic behavior of stationary mappings on $\mathbb{X}_+$ is the next result.[6]

Lemma 3.3
Under Assumptions 1 and 3, suppose Φ is stationary. Then, the fixed point $\bar{x} \in \mathbb{X}_+$ is a saddle, i.e., the Eigenvalues λ_1 and λ_2 of $D\Phi(\bar{x})$ are real and satisfy $0 < |\lambda_1| < 1 < |\lambda_2|$.

The stable manifold.
The stability result from Lemma 3.3 implies that the dynamics generated by a stationary

[6] One can show that saddle-path stability of interior steady states is a generic phenomenon of mappings of the form (11) even if Assumption 3 is not satisfied. For instance, if Φ has three hyperbolic fixed points $\bar{x}^{(i)} = (\bar{w}^{(i)}, \bar{b}^{(i)}) \in \mathbb{X}_+$, $i \in \{1, 2, 3\}$ where $\bar{w}^{(1)} < \bar{w}^{(2)} < \bar{w}^{(3)}$, both $\bar{x}^{(1)}$ and $\bar{x}^{(3)}$ are saddles while $\bar{x}^{(2)}$ is unstable, i.e., both Eigenvalues of $D\Phi(\bar{x}^{(2)})$ exceed unity in absolute value. The problem that arises with multiple steady states is that the stable manifold defined below can not be represented as the graph of a function $\mathcal{M}$ defined globally on $\mathbb{R}_{++}$ in this case.

map Φ display stable behavior only along a lower-dimensional subset of the state space. This subset is called the (globally) *stable manifold* $\mathbb{M}$ and consists of all initial points for which forward-iterates of the map Φ stay in $\mathbb{X}$ and converge to the steady state $\bar{x}$. Formally,

$$\mathbb{M} := \Big\{ x \in \mathbb{X} \,|\, \Phi^n(x) \in \mathbb{X} \; \forall n \geq 1 \, \wedge \lim_{n \to \infty} \Phi^n(x) = \bar{x} \Big\}. \tag{12}$$

The stable manifold $\mathbb{M}$ will play a key-role in the following sections. First note that $\mathbb{M} \subset \mathbb{X}_+$ by the second requirement in (12). Second, $\mathbb{M}$ is *self-supporting* under Φ, i.e., $\Phi(\mathbb{M}) \subset \mathbb{M}$. Third, as will be shown below, $\mathbb{M}$ separates initial points which are sustainable – in the sense defined above – from those which leave the state space $\mathbb{X}$ in finite time under iteration of Φ. This last property requires a geometric characterization of $\mathbb{M}$ as the graph of a strictly increasing C^1 function $\mathcal{M} : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$. For this purpose, we make the following additional assumption where we let $w_{\max} := \lim_{w \to \infty} \phi(w, 0)$ and $\mathbb{Y} :=]0, w_{\max}[\times \mathbb{R}_{++}$.

Assumption 4
Φ is a C^1-diffeomorphism between the open sets $\mathbb{X}_+$ and $\mathbb{Y}$.

The final result of this section provides the desired geometric characterization of the globally stable manifold $\mathbb{M}$ and the separation property mentioned above. The proof of (i) employs several ideas also used in Galor (1992).

Lemma 3.4
Under Assumptions 1, 2, 3, and 4, let Φ be stationary. Then, the following holds:

(i) There exists a C^1 function $\mathcal{M} : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$, $\mathcal{M}' > 0$ such that $\mathbb{M} = \mathrm{graph}\mathcal{M}$.

(ii) For any $x = (w, b) \in \mathbb{X}$, the following holds:

(a) If $b < \mathcal{M}(w)$, then $\Phi^t(x) \in \mathbb{X}$ for all $t \geq 0$ and $\lim_{t \to \infty} \Phi^t(x) = \bar{x}^0$.

(b) If $b = \mathcal{M}(w)$, then $\Phi^t(x) \in \mathbb{M}$ for all $t \geq 0$ and $\lim_{t \to \infty} \Phi^t(x) = \bar{x}$ monotonically.

(c) If $b > \mathcal{M}(w)$, then there exists $t_0 \geq 0$ such that $\Phi^{t_0}(x) \notin \mathbb{X}$.

Based on the characterization in (i), Lemma 3.4 (ii) shows that all states strictly below $\mathbb{M}$ converge to the bubbleless steady state under iteration of Φ while initial states on $\mathbb{M}$ converge to the bubbly steady state $\bar{x}$. All states above $\mathbb{M}$ are unsustainable and leave the state space in finite time. As a consequence, the set of sustainable states defined as

$$\overline{\mathbb{X}} := \Big\{ x \in \mathbb{X}_+ \,|\, \Phi^n(x) \in \mathbb{X}_+ \; \forall n \geq 0 \Big\} \tag{13}$$

is given by $\overline{\mathbb{X}} = \{(w, b) \in \mathbb{X}_+ | b \leq \mathcal{M}(w)\}$. Note from Lemma 3.4 (ii) that $\overline{\mathbb{X}}$ is self-supporting for Φ, i.e., $\Phi(\overline{\mathbb{X}}) \subset \overline{\mathbb{X}}$ and that no superset of $\overline{\mathbb{X}}$ can be self-supporting. Therefore, restricting Φ to this set permits to transform the pseudo-dynamical system (11) into a proper dynamical system. In the deterministic case, the findings from Lemma 3.4 generalize the results in Tirole (1985) whose dynamic structure constitutes a special case of the general class of mappings (11). Also note that $\overline{\mathbb{X}}$ defined in (13) is empty if Φ is expansive due to Lemma 3.2.

4 Existence of Bubbly Equilibria

The goal of this section is to exploit the dynamic properties of the equilibrium mappings to construct bubbly equilibria. In order to apply the results from the previous section, each of the equilibrium mappings $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ defined in (10a,b) has to satisfy the additional Assumptions 1 to 4. The first part of this section provides conditions under which this is the case. It should be noted, however, that the conditions to be presented are from necessary to obtain the desired properties. For this reason, the main results stated as Theorems 1 and 2 below employ the derived properties embodied in Assumptions 1 to 4 which may well be satisfied even if the conditions to be presented next are not.

Conditions for Assumption 1

Given $\varepsilon \in \mathcal{E}$, let $\phi(x) := \mathcal{W}(\mathcal{K}(x);\varepsilon)$ and $\psi(x) := \vartheta(\varepsilon)\mathcal{Z}(x)$, $x \in \mathbb{X}$ to observe that $\Phi(\cdot;\varepsilon)$ defined in (10a,b) has the structure assumed in (11). Under the hypotheses of Lemma 2.2, both mappings $\mathcal{K}$ and $\mathcal{Z}$ are strictly monotonic. Further, the properties of the production function f imply that $\mathcal{W}(\cdot;\varepsilon)$ is C^1, strictly monotonic, and satisfies $\lim_{k\searrow 0}\mathcal{W}(k;\varepsilon) = 0$. These observations lead to the following

Lemma 4.1
In addition to (T1), (U1), and (U2), suppose either $\vartheta = \mathrm{id}_{\mathcal{E}}$ or let (U3) hold. Then, each $\Phi(\cdot;\varepsilon)$ satisfies Assumption 1.

Conditions for Assumption 2

To obtain conditions under which a bubbleless steady state $\bar{x}^0_\varepsilon \in \mathbb{X}_0$ of $\Phi(\cdot;\varepsilon)$ exists, recall that the bubbleless equilibrium in our model coincides with the one in Wang (1993). He uses the condition $\lim_{w\to 0}\Phi^{(1)}_w(w,0;\varepsilon) > 1$ to ensure existence of a positive steady state. While this appears to be a standard restriction in the literature also imposed, e.g., in Hauenschild (2002), it does not guarantee that the steady state is unique. Therefore, the following result adds sufficient conditions under which uniqueness holds. As the return at the bubbleless steady state varies continuously with the parameters of the model, the additional requirement of a non-unit return from Lemma 3.2 should generically be satisfied.

Lemma 4.2
Under (T1), (T2), (U1), (U2), and (U4), each $\Phi(\cdot;\varepsilon)$ has at most one fixed point in $\mathbb{X}_0$. If, in addition, (U5) holds and $\lim_{w\to 0}\Phi^{(1)}_w(w,0;\varepsilon) > 1$, then $\Phi(\cdot;\varepsilon)$ satisfies Assumption 2.

The assumption of a unique bubbleless steady state is imposed throughout in Tirole (1985), Weil (1987), and almost any deterministic study of bubbles. In the stochastic case studied here, it will offer a convenient way to distinguish stationary versus expansive behavior of the equilibrium mappings using the result from Lemma 3.1. In addition, one can show that the existence of a bubbleless steady state of $\Phi(\cdot;\varepsilon)$ for each $\varepsilon \in \mathcal{E}$ is also necessary for bubbly equilibria to exist at all. To see this, note that if some $\Phi(\cdot;\varepsilon)$ failed to have a bubbleless steady state, the boundary behavior of f and Lemma 2.2 would imply $\Phi^{(1)}(w,b;\varepsilon) \leq \Phi^{(1)}(w,0;\varepsilon) < w$ for all $x = (w,b) \in \mathbb{X}$. Thus, the economy would impoverish under forward-iteration of $\Phi(\cdot;\varepsilon)$ in the sense that the wage and capital stock converge to zero. In this case, one can easily show that any initial state $x_0 \in \mathbb{X}_+$ will leave the state space $\mathbb{X}$ in finite time, i.e., the map $\Phi(\cdot;\varepsilon)$ will display expansive behavior in the exact same sense as defined in the previous section. As argued below, there can be no bubbly equilibria in this case.

Conditions for Assumption 3

In the deterministic case studied in Tirole (1985), there can be at most one bubbly steady state. Essentially, this is because the steady state interest on the bubble is directly pinned down by the growth rate of the economy. In the stochastic case studied here, a similar result holds if the bubble is capital-equivalent, i.e., $\vartheta = \mathrm{id}_{\mathcal{E}}$ in (6). If the returns on bubbles and capital exhibit a different risk structure, however, additional restrictions on the fundamentals of the economy stated are required to guarantee uniqueness of the bubbly steady state. Conditions under which this holds are stated next.

Lemma 4.3

In addition to (T1), (U1), and (U2), let either $\vartheta = \mathrm{id}_{\mathcal{E}}$ or (T2), (U3), and (U4) hold. Then $\Phi(\cdot;\varepsilon)$ satisfies Assumption 3, i.e., has at most one steady state in $\mathbb{X}_+$ which is hyperbolic.

Conditions for Assumption 4

In addition to the uniqueness condition from Assumption 3, the key property needed to construct a globally stable manifold as in Section 3 is that $\Phi(\cdot;\varepsilon)$ be a C^1-diffeomorphism. Our next result shows that this property requires little more than the restrictions imposed in Lemma 2.2. Here we define $w_{\max}(\varepsilon) := \lim_{k\to\infty} \mathcal{W}(k;\varepsilon)$ and $\mathbb{Y}_\varepsilon :=]0, w_{\max}(\varepsilon)[\times\mathbb{R}_{++}$.[7]

Lemma 4.4

In addition to (T1), (U1), and (U2), let (U5) and either $\vartheta = \mathrm{id}_{\mathcal{E}}$ or (U3) hold. Then $\Phi(\cdot;\varepsilon)$ satisfies Assumption 4, i.e., it is a C^1-diffeomorphism between the sets $\mathbb{X}_+$ and $\mathbb{Y}_\varepsilon$.

Necessary conditions for bubbly equilibria.

The remainder of this section assumes that each member of the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ satisfies Assumptions 1, 2, 3, and 4. For ease of exposition, we also assume that $\mathcal{E}$ is a finite set. Generalizations of this restriction are straightforward and discussed below.

A first observation based on the result from Lemma 3.2 is that existence of a bubbly equilibrium requires each mapping $\Phi(\cdot;\varepsilon)$ to be stationary. For if some member $\Phi(\cdot;\varepsilon')$, $\varepsilon' \in \mathcal{E}$ were expansive, any initial state $x_0 \in \mathbb{X}_+$ would leave the state space under forward-iteration of this mapping in finite time $t_0 \in \mathbb{N}$. Since the event of drawing $\varepsilon_t = \varepsilon'$ for all $1 \le t \le t_0$ occurs with positive probability $\nu(\{\varepsilon'\})^{t_0} > 0$, the equilibrium condition $x_t \in \mathbb{X}$ $\mathbb{P}$–almost surely for all $t \ge 0$ is clearly not satisfied in this case.

Therefore, invoking Lemma 3.1 the bubbly return at the bubbleless steady state $(\bar{w}^0_\varepsilon, 0)$ must be smaller than unity for each $\varepsilon \in \mathcal{E}$. This condition can be stated as

$$\max_{\varepsilon\in\mathcal{E}}\Big\{\mathcal{R}^\star(\mathcal{Z}(\bar{w}^0_\varepsilon, 0), \varepsilon)\Big\} < 1. \tag{14}$$

In the deterministic case $\mathcal{E} = \{\varepsilon\}$, (14) reduces to the existence condition in Tirole (1985). Solving (7a) for z, condition (14) may equivalently be written as

$$\min_{\varepsilon\in\mathcal{E}}\left\{\frac{\mathbb{E}_\nu[\vartheta(\cdot)v'(\bar{k}^0_\varepsilon\mathcal{R}(\bar{k}^0_\varepsilon;\cdot))]}{\vartheta(\varepsilon)u'(\bar{w}^0_\varepsilon - \bar{k}^0_\varepsilon)}\right\} > 1. \tag{15}$$

Here $\bar{k}^0_\varepsilon := \mathcal{K}(\bar{x}^0_\varepsilon)$ is the corresponding steady state capital stock. As the bubbleless steady state is independent of ϑ, (15) may be seen as a restriction on the risk-structure of the bubble. In particular, this condition is invariant to re-scaling the function ϑ.

[7] This is consistent with the definition of $w_{\max}$ in Assumption 4 as $\lim_{w\to\infty}\mathcal{K}(w,0) = \infty$ under (U5).

A second observation is that restrictions on the initial state $x_0 = (w_0, b_0)$ are required. To this end, let (15) hold. Then, each $\Phi(\cdot;\varepsilon)$ is stationary and, therefore, has a bubbly steady state $\bar{x}_\varepsilon = (\bar{w}_\varepsilon, \bar{b}_\varepsilon) \in \mathbb{X}_+$ which is unique by Assumption 3. Let $\mathbb{M}_\varepsilon$ be the associated stable manifold defined as in (12). Then, Assumption 4 and Lemma 3.4 (i) permit to represent each $\mathbb{M}_\varepsilon$ as the graph of an increasing C^1 function $\mathcal{M}_\varepsilon : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$. By Lemma 3.4 (ii), it is clear that the initial state $x_0 = (w_0, b_0)$ and, in fact, any successive state x_t must lie below each $\mathbb{M}_\varepsilon$, $\varepsilon \in \mathcal{E}$. Thus, define for each $w > 0$ the critical value

$$\mathcal{M}^{\mathrm{crit}}(w) := \min_{\varepsilon \in \mathcal{E}} \{\mathcal{M}_\varepsilon(w). \tag{16}$$

Note that $\mathcal{M}^{\mathrm{crit}}$ is well-defined as the minimum is taken over finitely many values in $\mathcal{E}$. Further, $\mathcal{M}^{\mathrm{crit}}$ is continuous and strictly increasing although not necessarily differentiable. The curve $w \mapsto \mathcal{M}^{\mathrm{crit}}(w)$, $w > 0$ defines the boundary of the set of points which lie below each of the stable sets $\mathbb{M}_\varepsilon$ defined in (12) for all $\varepsilon \in \mathcal{E}$ and it follows immediately from Lemma 3.4 (ii) that any equilibrium process must take values in this set.

Combining the previous insights, we are now in a position to state our first main result which provides necessary conditions for bubbly equilibria to exist.

Theorem 1
Suppose $\mathcal{E}$ is finite. Let $\Phi(\cdot;\varepsilon)$ defined in (10a,b) satisfy Assumptions 1, 2, 3, and 4 for each $\varepsilon \in \mathcal{E}$. Then, the existence of a bubbly equilibrium requires the following conditions:

(i) For each $\varepsilon \in \mathcal{E}$, $\Phi(\cdot;\varepsilon)$ is stationary, i.e., condition (15) holds.

(ii) The initial state (w_0, b_0) satisfies $0 < b_0 \leq b_0^{\mathrm{crit}} := \mathcal{M}^{\mathrm{crit}}(w_0)$ defined as in (16).

For the deterministic case, Theorem 1 completely recovers the results in Tirole (1985). His setup corresponds to the special case where $\nu = \delta_\varepsilon$ is a Dirac measure concentrated at some point $\varepsilon > 0$, i.e., $\mathcal{E} = \{\varepsilon\}$. In this case, the condition (ii) in Theorem 1 is also sufficient and each $b_0 \leq b_0^{\mathrm{crit}}$ defines a bubbly equilibrium.
In the general stochastic case, however, the conditions in Theorem 1 may not be sufficient. To see this, suppose the initial state $x_0 = (w_0, b_0) \in \mathbb{X}_+$ satisfies $b_0 \leq \mathcal{M}^{\mathrm{crit}}(w_0)$. Then, by (16) $b_0 \leq \mathcal{M}_\varepsilon(w_0)$ for all $\varepsilon \in \mathcal{E}$. It follows from Lemma 3.4 (ii) that for any constant sequence $(\varepsilon, \varepsilon, \ldots)$ where $\varepsilon \in \mathcal{E}$ the sequence of states $x_t := \Phi(x_{t-1}, \varepsilon)$, $t \geq 0$ satisfies $b_t \leq \mathcal{M}_\varepsilon(w_t)$ for all $t \geq 0$ and converges to the bubbleless steady state $\bar{x}^0_\varepsilon$ if $b_0 < \mathcal{M}_\varepsilon(w_0)$ and to the bubbly steady state $\bar{x}_\varepsilon$ otherwise. However, this convergence may be non-monotonic, i.e., it can happen that for some $\varepsilon' \in \mathcal{E}$ for which $\mathcal{M}_{\varepsilon'} \neq \mathcal{M}^{\mathrm{crit}}$, the sequence $x'_t := \Phi^t(x_0; \varepsilon')$, $t \geq 0$ temporarily exceeds the graph of $\mathcal{M}^{\mathrm{crit}}$, as indicated by the dashed arrows in Figure 1. Suppose this happens after t_0 periods, i.e., $b'_{t_0} > \mathcal{M}^{\mathrm{crit}}(w'_{t_0})$. Let $\varepsilon'' \in \mathcal{E}$ be the value for which $\mathcal{M}^{\mathrm{crit}}(w'_{t_0}) = \mathcal{M}_{\varepsilon''}(w'_{t_0})$. Then, $b'_{t_0} > \mathcal{M}_{\varepsilon''}(w'_{t_0})$ and it follows from Lemma 3.4 (ii) that there exists a finite time $t_1 \in \mathbb{N}$ for which $\Phi^{t_1}(x'_{t_0}; \varepsilon'') \notin \mathbb{X}$. As the event of drawing $\varepsilon_t = \varepsilon'$ for $t = 1, \ldots, t_0$ and $\varepsilon_t = \varepsilon''$ for $t = t_0 + 1, \ldots, t_1$ has positive probability $\nu(\{\varepsilon'\})^{t_0} \nu(\{\varepsilon''\})^{t_1 - t_0}$, the initial choice x_0 is not compatible with an equilibrium. Conclude from this that, in general, the value defined in (16) is only an upper bound for the initial bubble b_0. Also note that the previous arguments become obsolete if each $\mathcal{M}_\varepsilon$ is independent of ε, a case which holds in the example studied below.

A direct consequence of the previous observations is that the following additional property is required for the conditions in Theorem 1 to be sufficient: The set of points below the graph of $\mathcal{M}^{\text{crit}}$ must be self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$. Formally,

$$\forall w > 0: \; b \leq \mathcal{M}^{\text{crit}}(w) \quad \Rightarrow \quad \Phi^{(2)}(w,b;\varepsilon) \leq \mathcal{M}^{\text{crit}}(\Phi^{(1)}(w,b;\varepsilon)) \quad \forall \varepsilon \in \mathcal{E}. \tag{17}$$

The additional condition (17) leads to the following corollary.

Corollary 1
Under the hypotheses of Theorem 1, suppose conditions (15) and (17) hold. Then, each (w_0, b_0) for which $0 < b_0 \leq \mathcal{M}^{\text{crit}}(w_0)$ defines a bubbly equilibrium.

An alternative interpretation of (17) can be obtained by defining for each $\Phi(\cdot;\varepsilon)$ the set of sustainable states $\overline{\mathbb{X}}_\varepsilon$ as in (13). Then, as demonstrated above, the fact that each $\overline{\mathbb{X}}_\varepsilon$ is self-supporting for $\Phi(\cdot;\varepsilon)$ *does not imply* that the intersection $\overline{\mathbb{X}} := \cap_{\varepsilon\in\mathcal{E}}\overline{\mathbb{X}}_\varepsilon$ is self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$. As $\overline{\mathbb{X}} = \{(w,b) \in \mathbb{X}_+ \mid b \leq \mathcal{M}^{\text{crit}}(w)\}$, this is precisely what is ensured by the additional condition (17) under which any $x_0 \in \overline{\mathbb{X}}$ defines a bubbly equilibrium. Also recall from Section 3 that $\overline{\mathbb{X}}_\varepsilon$ and, therefore, $\overline{\mathbb{X}}$ would be empty if some map $\Phi(\cdot;\varepsilon)$ were expansive.

Figure 1 illustrates the previous insights for the case with two shocks where $\mathcal{E} = \{\varepsilon', \varepsilon''\}$. The dashed arrows represent the case which is excluded by (17).

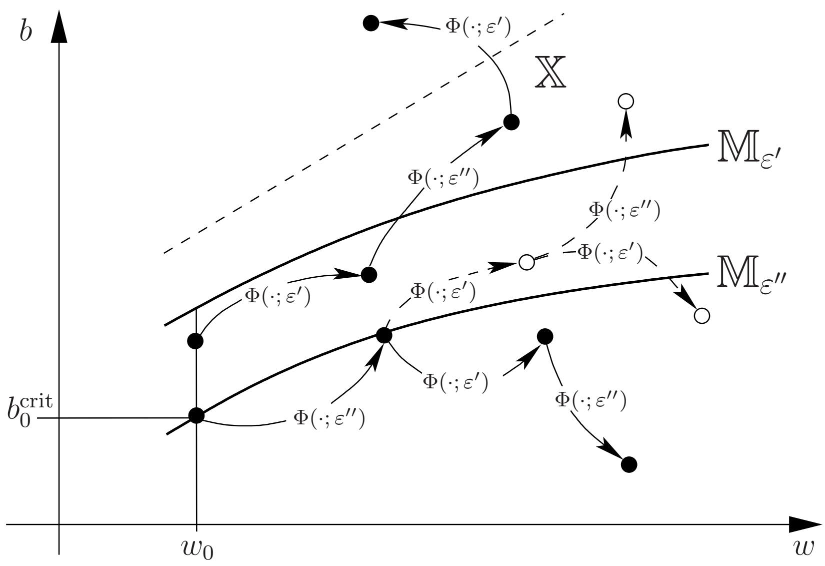

Figure 1: Dynamics generated by mixing two stationary mappings.

Sufficient conditions for bubbly equilibria.
As condition (17) is not stated in terms of the primitives of the model, it is not clear which restrictions it imposes on the economy $\mathscr{E}$ and whether it can be satisfied at all. As our second main result, we now establish that (17) holds automatically if the bubble is riskless, i.e., if ϑ in (6) is a constant function $\bar{\vartheta} > 0$. In this case, the existence condition (15) reads

$$\min_{\varepsilon\in\mathcal{E}} \left\{ \frac{\mathbb{E}_\nu[v'(\bar{k}^0_\varepsilon \mathcal{R}(\bar{k}^0_\varepsilon;\cdot))]}{u'(\bar{w}^0_\varepsilon - \bar{k}^0_\varepsilon)} \right\} > 1. \tag{18}$$

Observe the similarity of (18) to the existence conditions (2) and (3) derived in Manuelli (1990, p.273) for a stochastic exchange economy. For the case with a riskless bubble, we now have the following additional properties of the mappings $\mathcal{M}_\varepsilon$ which characterize the stable sets $\mathbb{M}_\varepsilon$. Note that the result does not require finiteness of $\mathcal{E}$.

Lemma 4.5
Let each $\Phi(\cdot;\varepsilon)$ be stationary and satisfy Assumptions 1, 2, 3, and 4. If $\vartheta \equiv \bar{\vartheta}$, then $\varepsilon < \varepsilon'$ implies $\mathcal{M}_\varepsilon(w) < \mathcal{M}_{\varepsilon'}(w)$ for all $w > 0$, $\varepsilon, \varepsilon' \in \mathcal{E}$. Moreover, $\mathcal{M}^{\text{crit}} = \mathcal{M}_{\varepsilon_{\min}}$ satisfies (17).

Lemma 4.5 states that for a riskless bubble, the map $\varepsilon \longmapsto \mathcal{M}_\varepsilon(w)$ is strictly increasing on $\mathcal{E}$ for all $w > 0$. In particular, $\varepsilon \neq \varepsilon'$ implies $\mathbb{M}_\varepsilon \cap \mathbb{M}_{\varepsilon'} = \emptyset$, i.e., the stable sets pertaining to different shocks have an empty intersection, a property which will become important in the next paragraph. Using the insights from Lemma 4.5, we are in a position to state our second main result.

Theorem 2
Let $\mathcal{E}$ be finite and each $\Phi(\cdot;\varepsilon)$ defined in (10a,b) satisfy Assumptions 1, 2, 3, and 4. If $\vartheta \equiv \bar{\vartheta} > 0$ and condition (18) holds, each $0 < b_0 \leq \mathcal{M}_{\varepsilon_{\min}}(w_0)$ defines a bubbly equilibrium.

Temporary nature of stochastic bubbles.
While bubbly equilibria exist under the conditions (15) and (17), generically these bubbles are only temporary and converge to zero with probability one. Unlike the case in Tirole (1985), this holds even if $b_0 = \mathcal{M}^{\text{crit}}(w_0)$. Structurally, the reason is that positive *stable sets* of the dynamics (10a,b), i.e., compact subsets $\mathbb{A} \subset \mathbb{X}_+$ which are self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ such that $\Phi(\mathbb{A};\varepsilon) \subset \mathbb{A}$ for all $\varepsilon \in \mathcal{E}$ typically fail to exist. To see this, note from Lemma 3.4 that $\mathbb{A} \subset \mathbb{X}_+$ closed and self-supporting under $\Phi(\cdot;\varepsilon)$ requires $\mathbb{A} \subset \mathbb{M}_\varepsilon$. Hence, positive stable sets are subsets of $\cap_{\varepsilon\in\mathcal{E}}\mathbb{M}_\varepsilon$ which is typically empty. In particular, as shown in Lemma 4.5 this is true if the bubbly asset is riskless, i.e., $\vartheta \equiv \bar{\vartheta} > 0$. In this case, all equilibria will be asymptotically bubbleless with probability one, i.e., $\lim_{t\to\infty} b_t = 0$ $\mathbb{P}$-a.s.
This last finding entails serious consequences for the discussion in Bertocchi (1994) about the existence of stable sets in a similar model with the bubble corresponding to riskless government debt. Referring to the equilibrium scenarios discussed there, Lemma 3.3 already showed that bubbly steady states which are asymptotically stable and would give rise to stable sets with positive bubbles do not exist. Lemma 4.5 now shows that such stable sets are directly excluded by the assumption that debt offers a riskless return.

An example with persistent bubbles.
The following example, however, shows that stable sets giving rise to persistent bubbles may exist in certain situations where the return on the bubble is risky. Let $U(c^y, c^o) = (1-\gamma)\ln c^y + \gamma \ln c^o$, $0 < \gamma < 1$ and $f(k) = k^\alpha$, $0 < \alpha < 1$. This parametrization is widely used in many deterministic studies, cf. Michel & Wigniolle (2003), or Kunieda (2008). As condition (U2) is violated in this case, the tighter restriction $b_t \leq \gamma w_t$ is required to ensure that a solution to (8) exists. Thus, redefine the endogenous state space

$$\mathbb{X}' = \{(w, b) \in \mathbb{R}^2_+ \,|\, b < \gamma w\}. \tag{19}$$

Suppose the bubble is capital-equivalent, i.e., $\vartheta = \mathrm{id}_{\mathcal{E}}$. Solving (8) using (2a,b), the equilibrium mapping defined as in (10a,b) takes the explicit form $\Phi : \mathbb{X}' \times \mathcal{E} \longrightarrow \mathbb{R}^2$,

$$w_{t+1} = \Phi^{(1)}(w_t, b_t; \varepsilon_{t+1}) \;:=\; \varepsilon_{t+1}(1-\alpha)(\gamma w_t - b_t)^{\alpha} \tag{20a}$$

$$b_{t+1} = \Phi^{(2)}(w_t, b_t; \varepsilon_{t+1}) \;:=\; \varepsilon_{t+1}\alpha(\gamma w_t - b_t)^{\alpha-1} b_t. \tag{20b}$$

By direct computations, one verifies that $\Phi(\cdot;\varepsilon)$ satisfies Assumptions 1 to 4 for each $\varepsilon \in \mathcal{E}$ such that all the results from Section 3 extend to the present case with the modified state space given by (19).[8] For each $\varepsilon \in \mathcal{E}$ the unique bubbleless steady state $(w^0_\varepsilon, 0)$ can be computed explicitly as $\bar{w}^0_\varepsilon = (\varepsilon(1-\alpha)\gamma^\alpha)^{1/(1-\alpha)}$ and the associated ex-post return on capital and the bubble is $\mathcal{R}(\mathcal{K}(w^0_\varepsilon, 0); \varepsilon) = \gamma^{-1}\alpha/(1-\alpha)$, $\varepsilon \in \mathcal{E}$. The latter determines whether each equilibrium mapping is stationary or expansive. This leads to the following result.

Lemma 4.6
Given $\varepsilon \in \mathcal{E}$, define $\Phi(\cdot;\varepsilon)$ as in (20a,b). If $\bar{\beta} := \gamma - \frac{\alpha}{1-\alpha} > 0$, then the following holds:

(i) $\Phi(\cdot;\varepsilon)$ is stationary and has a unique steady state $\bar{x}_\varepsilon \in \mathbb{X}_+$ which is a saddle.

(ii) The sets $\mathbb{M}_\varepsilon$ defined as in (12) take the form $\mathbb{M}_\varepsilon \equiv \mathbb{M} := \big\{(w,b) \in \mathbb{R}^2_{++} \,\big|\, b = \bar{\beta} w\big\}$.

Lemma 4.6 (ii) shows that in this particular case, the sets $\mathbb{M}_\varepsilon$ defined as in (12) are independent of ε. Thus, one can show by direct computations that states below $\mathbb{M}$ remain below this set, i.e., condition (17) is satisfied. This leads to the following result.

Theorem 3
For the previous parametrization, suppose $\bar{\beta} > 0$. Then, each $x_0 = (w_0, b_0) \in \mathbb{X}'$ for which $b_0 \leq \bar{\beta} w_0$ defines a bubbly equilibrium where the bubble is capital-equivalent, i.e., $\vartheta = \mathrm{id}_{\mathcal{E}}$.

The key feature of this example is that the set $\mathbb{M} = \cap_{\varepsilon\in\mathcal{E}}\mathbb{M}_\varepsilon$ is self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$. Thus, whenever $x_0 \in \mathbb{M}$, the state process $\{x_t\}_{t\geq 0}$ generated by (20a,b) stays in $\mathbb{M}$ for all t. Moreover, the state dynamics converge to a compact subset of $\mathbb{M}$ defined by the bubbly fixed points $((\bar{w}_\varepsilon, \bar{b}_\varepsilon))_{\varepsilon\in\mathcal{E}}$ of the mappings $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ which is a stable set. Thus, in this special case, setting the bubble b_0 equal to its maximum value $b_0^{\mathrm{crit}} = \bar{\beta} w_0$ yields a result similar to the deterministic case in Tirole (1985) where the bubble fails to die out and in fact converges to a positive stable subset of the state space.

The final part of this section outlines some extensions to which the previous setup should be amendable.

Infinite shock spaces
It is straightforward to extend the results from Theorems 1 and 2 to the case with an infinite shock space, e.g., where $\mathcal{E} = [\varepsilon_{\min}, \varepsilon_{\max}]$. In this case, define the sets

$$\mathcal{E}^s := \{\varepsilon \in \mathcal{E} \,|\, \mathcal{R}^\star(\mathcal{Z}(\bar{w}^0_\varepsilon, 0), \varepsilon) < 1\} \tag{21}$$

and $\mathcal{E}^x := \mathcal{E}\setminus\mathcal{E}^s$. As $\mathcal{R}^\star$ from (6) is Caratheodory and the bubbleless steady state $\bar{x}^0_\varepsilon$ varies continuously with ε by the Implicit Function Theorem, both sets $\mathcal{E}^s$ and $\mathcal{E}^x$ are measurable.

[8] As the state space is now given by (19), the boundary properties in Assumption 1 must be restated as $\lim_{\gamma w - b\searrow 0}\phi(w,b) = 0$ and $\lim_{\gamma w - b\searrow 0}\psi(w,b) = \infty$. All arguments which rely on this boundary behavior, e.g., the proofs of Lemma 3.1 or Lemma 3.4, must (and can easily) be adapted accordingly.

They represent shocks associated with drawing a stationary respectively expansive mapping $\Phi(\cdot;\varepsilon)$. Extending the arguments developed above, the existence of a bubbly equilibrium requires $\nu(\mathcal{E}^x) = 0$, i.e., the probability of drawing an expansive map must be zero. In addition, an upper bound on initial conditions must be established, which is obtained by replacing (16) by $\mathcal{M}^{\text{crit}}(w) := \inf_{\varepsilon\in\mathcal{E}^s}\{\mathcal{M}_\varepsilon(w)\}$. In particular, if ϑ is continuous, e.g., if the bubble is risk-less or capital-equivalent, and $\mathcal{E}^s$ is compact, all previous conditions and the results stated in Theorems 1 and 2 remain valid if in (14) to (18) $\mathcal{E}$ is replaced by $\mathcal{E}^s$.

Bubbles with state-dependent risk-structure.
A key restriction imposed throughout the previous analysis is that the risk-structure of the bubbly asset is time invariant. A natural and interesting extension would be to consider bubbles with a risk structure that varies with the current endogenous state of the economy. Formally, one would replace (6) by an arbitrary measurable or even continuous function $\vartheta : \mathcal{E} \times \mathbb{X} \to \mathbb{R}_{++}$ such that

$$r^\star_{t+1} = \vartheta(\varepsilon_{t+1}; x_t)z_t. \tag{22}$$

Maintaining the hypotheses of Lemma 2.1, one observes that the entire equilibrium structure derived in Section 2 along with the state space definition (9) continue to hold under this modification. In particular, bubbly equilibria are generated by a family of dynamic mappings $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ and the existence of such equilibria requires each member of this family to generate bounded dynamics on a non-empty subset of $\mathbb{X}_+$. Further, the dynamic properties of the equilibrium mappings can be studied with the same techniques applied above as long as the map $\vartheta(\varepsilon;\cdot)$ is continuously differentiable. Apart from that, there seems to be considerable freedom in the form (22) and the key question is whether certain specifications change the monotonicity properties stated in Lemma 2.2 and, therefore, the qualitative dynamic properties derived in Section 3. In this regard, first numerical experiments indicate that for certain specifications some equilibrium mappings may even possess bubbly steady states which are asymptotically stable. Having said this, at least some equilibrium mappings should continue to display the saddle-path stability which is crucial for the construction of bubbly equilibria in this paper. This suspicion is supported by the observation that the previous modification has no implications whatsoever in the deterministic case where $\nu = \delta_{\bar\varepsilon}$. In any case, the basic approach to construct bubbly equilibria employed in this paper should remain fully applicable under this extension. A particularly intriguing question is whether the function ϑ in (22) can be chosen such that a positive stable set of the state dynamics exists and the bubble becomes persistent, as in the example from Lemma 4.6. Using different techniques from functional analysis, this issue is further explored in Barbie & Hillebrand (2014).

Stochastically bursting bubbles.
The previous structure can also be generalized to study bubbles which burst stochastically as in Weil (1987). In this case, let $\{\theta_t\}_{t\geq 0}$ be a sequence of i.i.d. random variables which, for simplicity, are also independent of the production shocks and take values in $\Theta := \{0,1\}$. Then, the shock at time t is now given by the random variable $\xi_t := (\varepsilon_t, \theta_t)$ with values in $\Xi := \mathcal{E} \times \Theta$. Consequently, the ex-ante bubble return takes the generalized form

$$r^\star_{t+1} = \mathcal{R}^\star(z_t;\xi_{t+1}) := \theta_{t+1}\vartheta(\varepsilon_{t+1})\, z_t. \tag{23}$$

In particular, the function ϑ in (23) can be chosen constant in which case $r^\star_{t+1}$ becomes independent of the fundamental shock ε_{t+1}. It is now straightforward to modify the Euler equations (8,b) and to determine z_t and k_{t+1} as functions of the current state $x_t = (w_t, b_t)$.

Then, bubbly equilibria are generated by randomly mixing the family $(\Phi(\cdot;\xi))_{\xi\in\Xi}$ where some equilibrium mappings $\Phi(\cdot;\xi) : \mathbb{X} \longrightarrow \mathbb{R}^2_+$ now map bubbly states $x_t \in \mathbb{X}_+$ into bubbleless states $x_{t+1} \in \mathbb{X}_0$, i.e., the bubble 'bursts' whenever $\xi = (\varepsilon, 0)$. Clearly, these latter mappings trivially generate 'stationary dynamic behavior' in the sense that each state $x \in \mathbb{X}_+$ is sustainable under forward-iteration of $\Phi(\cdot;\xi)$. One can now repeat the entire dynamic analysis from the previous sections to obtain necessary and sufficient conditions for bubbly equilibria to exist in such an extended setup. Moreover, by varying the set Θ and its interpretation the generalized form (23) would also permit to incorporate 'extrinsic uncertainty' such as sunspots in the analysis.

Broader classes of economies
The setup in Wang (1993) has been extended in various directions to include non-additive utility, correlated production shocks, and more general, so-called non-classical production functions. Recent examples may be found in Morand & Reffett (2007), McGovern et al. (2013), or Hillebrand (2014). In principle, it should be possible to extend the study of the present paper to these more general classes of economies as long as the bubbleless equilibrium is unique and the equilibrium mappings are smooth. The latter is required in order to apply the methods used in this paper which made repeated use of the implicit function theorem and the stable manifold theorem. A large class of economies having this structure is identified in Hillebrand (2014).

5 Bubbles with Borrowing Constraints

In the frictionless economy studied in Tirole (1985), bubbly equilibria only exist if the bubbleless equilibrium suffers from overaccumulation of capital. To explain the emergence of asset bubbles in the presence of underaccumulation, several approaches in the literature study deterministic OLG economies with frictions such as cash-in advance constraints in Michel & Wigniolle (2003) or borrowing constraints in Kunieda (2008). The present section extends the setup from Kunieda (2008) to show that his findings carry over to a stochastic environment as well.

Heterogeneous consumers.
Following Kunieda (2008) , we modify the previous OLG structure by assuming that each generation now consists of a continuum of heterogeneous consumers with index set $\Lambda := [\lambda_{\min}, \lambda_{\max}]$ where $0 < \lambda_{\min} < 1 < \lambda_{\max}$. A consumer born at time $t \geq 0$ is identified by her investment productivity $\lambda \in \Lambda$ which determines the amount of capital obtained by each consumption good invested at time t. Specifically, if consumer $\lambda \in \Lambda$ invests $s_t \geq 0$ units at time t, she owns λs_t units of productive capital at time $t+1$. The productivity index λ is continuously distributed on the interval Λ. The distribution function $G : \Lambda \longrightarrow [0,1]$ has a continuous density function $g : \Lambda \longrightarrow \mathbb{R}_{++}$ with respect to Lebesgue measure on Λ. Assuming $\mathbb{E}[\lambda] = \int_\Lambda lg(l)d\,l = 1$, the earlier setup is recovered 'on average'.

The following analysis restricts attention to the parametrization employed in Kunieda (2008) with log-additive utility $U(c^y, c^o) = (1-\gamma)\log c^y + \gamma \log c^o$, $0 < \gamma < 1$ and Cobb-Douglas production $f(k) = k^\alpha$, $0 < \alpha < 1$. Given labor income $w_t > 0$ and the returns on capital and bubbles, the decision problem faced by consumer $\lambda \in \Lambda$ reads:

$$\max_{b,s}\Big\{(1-\gamma)\ln(w_t - b - s) + \gamma \mathbb{E}_t\big[\ln\big(r^\star_{t+1}\,b + r_{t+1}\,\lambda\,s\big)\big]\,\Big|\, s \geq 0, b \geq 0, b+s \leq w_t\Big\}. \quad (24)$$

Note that short-selling of the bubbly asset is no longer possible which is where the capital market imperfection enters. For simplicity, suppose that the return on the bubbly asset determined by (6) has the same risk-structure as capital, i.e., $\vartheta = \mathrm{id}_{\mathcal{E}}$ and $r^{\star}_{t+1} = \varepsilon_{t+1} z_t$ with z_t determined at time t. However, unlike the scenario from Section 2, it need not be the case that $z_t = f'(k_{t+1})$ at equilibrium since the per-unit return on capital investment s_t undertaken by consumer $\lambda \in \Lambda$ is now $\lambda r_{t+1} = \lambda \varepsilon_{t+1} f'(k_{t+1})$. Letting $\lambda_t := z_t / f'(k_{t+1})$, one infers from (24) that consumer λ will invest only in capital if $\lambda > \lambda_t$ and only in the bubble if $\lambda < \lambda_t$. Thus, direct calculations reveal that the unique solution to (24) is determined by the pair of demand functions[9]

$$s^{\lambda}_t = \mathcal{S}(\lambda; w_t, \lambda_t) \quad := \quad \gamma w_t \, 1_{[\lambda_{\min}, \lambda_t]}(\lambda) \tag{25a}$$

$$b^{\lambda}_t = \mathcal{B}(\lambda; w_t, \lambda_t) \quad := \quad \gamma w_t \, 1_{]\lambda_t, \lambda_{\max}]}(\lambda). \tag{25b}$$

Here, 1_A is the characteristic function of A, i.e., $1_A(x) = 1$ iff $x \in A$ and $1_A(x) = 0$ otherwise.

Recursive equilibrium structure.
Based on individual demands (25a,b), consider an arbitrary period $t \geq 0$. Defining $\mathbb{X}'$ as in (19), let $(w_t, b_t) \in \mathbb{X}'$ be given. The values z_t and k_{t+1} are determined such that the bubble is absorbed and next period's capital stock is consistent with individual savings. Using (25a,b), these conditions read

$$b_t = \int_{\Lambda} \mathcal{B}(\lambda; w_t, \lambda_t) h(\lambda) d\lambda \quad = \quad \gamma w_t \, G(\lambda_t) \tag{26a}$$

$$k_{t+1} = \int_{\Lambda} \lambda \mathcal{S}(\lambda; w_t, \lambda_t) h(\lambda) d\lambda \quad = \quad \gamma w_t \, \kappa(\lambda_t). \tag{26b}$$

Here we define $\kappa : \Lambda \longrightarrow [0,1]$, $\kappa(\lambda_t) := \int_{\lambda_t}^{\lambda_{\max}} \lambda g(\lambda) d\lambda$ which is strictly decreasing with boundary behavior $\kappa(\lambda_{\min}) = \mathbb{E}[\lambda] = 1$ and $\kappa(\lambda_{\max}) = 0$. As G is invertible, the first condition (26a) defines the equilibrium value λ_t as a map $L : [0, \gamma] \longrightarrow \Lambda$,

$$\lambda_t = L\left(\frac{b_t}{w_t}\right) := G^{-1}\left(\frac{1}{\gamma}\frac{b_t}{w_t}\right). \tag{27}$$

Note that L is strictly increasing with $L(0) = G^{-1}(0) = \lambda_{\min}$ and $L(\gamma) = G^{-1}(1) = \lambda_{\max}$. Using (27) in (26b) and the definition of λ_t, the values k_{t+1} and z_t are determined as

$$k_{t+1} = \mathcal{K}(w_t, b_t) \quad := \quad \gamma w_t \, \kappa\left(L\left(\frac{b_t}{w_t}\right)\right) \tag{28a}$$

$$z_t = \mathcal{Z}(w_t, b_t) \quad := \quad f'\big(\mathcal{K}(w_t, b_t)\big) L\left(\frac{b_t}{w_t}\right). \tag{28b}$$

Equilibrium dynamics.
Using (2a), (5), and (28a,b), the dynamics are generated by $\Phi = (\Phi^{(1)}, \Phi^{(2)}) : \mathbb{X}' \times \mathcal{E} \longrightarrow \mathbb{R}^2_+$

$$w_{t+1} = \Phi^{(1)}(w_t, b_t; \varepsilon_{t+1}) \quad := \quad \varepsilon_{t+1} \, (1 - \alpha) \, \big(\mathcal{K}(w_t, b_t)\big)^{\alpha} \tag{29a}$$

$$b_{t+1} = \Phi^{(2)}(w_t, b_t; \varepsilon_{t+1}) \quad := \quad \varepsilon_{t+1} \, \alpha \, \big(\mathcal{K}(w_t, b_t)\big)^{\alpha - 1} L\left(\frac{b_t}{w_t}\right) b_t. \tag{29b}$$

[9] It is arbitrarily assumed that the consumer invests only in capital if $\lambda = \lambda_t$. Since the set of consumers who have $\lambda = \lambda_t$ has measure zero, this assumption is irrelevant.

As in the example from the previous section, one verifies that $\Phi(\cdot;\varepsilon)$ defined in (29a,b) satisfies Assumptions 1 to 4 for each $\varepsilon \in \mathcal{E}$ with the modified state space given by (19). In particular, the definitions of L and κ and (28a) yield $\Phi^{(1)}(w,0;\varepsilon) = \varepsilon(1-\alpha)(\gamma w)^{\alpha}$. Thus, the dynamics (29a,b) coincide with (20a,b) along the bubbleless equilibrium. In particular, a unique bubbleless steady state $(\bar{w}^0_\varepsilon, 0)$ exists for each $\varepsilon \in \mathcal{E}$ where $\bar{w}^0_\varepsilon$ is defined as in the previous section. However, while the capital return at the bubbleless steady state continues to be $\mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon,0);\varepsilon) = \gamma^{-1}\alpha/(1-\alpha)$, the ex-post return on the bubble is now given by

$$\mathcal{R}^\star(\mathcal{Z}(\bar{w}^0_\varepsilon,0);\varepsilon) = \varepsilon\mathcal{Z}(\bar{w}^0_\varepsilon,0) = \frac{\lambda^{\min}}{\gamma}\frac{\alpha}{1-\alpha}. \tag{30}$$

Analogously to Section 4, the returns (30) are key for the dynamic properties of the mappings $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ in (29a ,b). In particular, the existence of a bubbly equilibrium requires that each $\Phi(\cdot;\varepsilon)$ be stationary, which is the case iff $\mathcal{R}^\star(\mathcal{Z}(\bar{w}^0_\varepsilon,0);\varepsilon) < 1$. Based on (30), we have the following result:

Lemma 5.1
Given $\varepsilon \in \mathcal{E}$, define $\Phi(\cdot;\varepsilon)$ as in (29a,b) and let $\frac{\lambda^{\min}}{\gamma}\frac{\alpha}{1-\alpha} < 1$. Then, the following holds:

(i) $\Phi(\cdot;\varepsilon)$ is stationary and has a unique steady state $\bar{x}_\varepsilon \in \mathbb{X}_+$ which is a saddle.

(ii) The sets $\mathbb{M}_\varepsilon$ defined as in (12) are of the form $\mathbb{M}_\varepsilon \equiv \mathbb{M} := \{(w,b) \in \mathbb{X} \,|\, b = \bar{\beta}w\}$.

Here, $\bar{\beta} > 0$ is the unique solution to $L(\beta) = \frac{1-\alpha}{\alpha}\gamma\kappa(L(\beta))$.

An immediate consequence of Lemma 5.1 (ii) is that a condition similar to (17) holds. This leads to the following main result of this section.

Theorem 4
For the previous parametrization, suppose $\frac{\lambda^{\min}}{\gamma}\frac{\alpha}{1-\alpha} < 1$ and define $\bar{\beta}$ as above. Then, each $(w_0, b_0) \in \mathbb{X}'$ for which $b_0 \le \bar{\beta}w_0$ defines an equilibrium with capital-equivalent bubble.

The previous extension with borrowing constraints preserves the essential dynamic features of the frictionless example from Section 4. In the present case, however, a sufficiently small value $\lambda_{\min}$ ensures that $\frac{\lambda^{\min}}{\gamma}\frac{\alpha}{1-\alpha} < 1$ and a bubbly equilibrium exists even if the steady state capital return exceeds unity, i.e., if the bubbleless equilibrium does not suffer from overaccumulation. One also observes that $\mathbb{M} := \cap_{\varepsilon\in\mathcal{E}}\mathbb{M}_\varepsilon$ is again self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ from (29a, b). Thus, whenever $x_0 \in \mathbb{M}$, the dynamics converge to a compact stable set. While bubbles are persistent in this particular case, we suspect that this persistence property should generically fail to hold as the analysis is extended to more general preferences and technologies, just as in the absence of frictions.

6 Conclusions

The previous analysis derived necessary and sufficient conditions under which bubbly equilibria exist in a frictionless OLG economy with random production and endogenous capital accumulation. A maximum sustainable bubble was identified which places an upper bound

on the initial condition extending the results for deterministic models in Tirole (1985). Unlike the deterministic case, however, bubbles in stochastic OLG models are generically non-persistent and vanish asymptotically with probability one even if the initial bubble is set to its maximum value. Introducing frictions such as borrowing constraints allows for bubbles to emerge even if the bubbleless equilibrium has overaccumulation of capital.
This last result was demonstrated for a particular parametrization of the model which is widely used in the literature. An interesting topic of future research might be to explore how this generalizes to the broader setup employed in the earlier chapters of this paper. Several other extensions of the model were already discussed in Section 4. A final set of questions concerns the welfare implications of bubbles and whether the conditions under which bubbly equilibria exist imply that the bubbleless equilibrium is inefficient. These and related questions are explored in Barbie & Hillebrand (2014).

A Mathematical Appendix

A.1 Proof of Lemma 2.1

Given $(w,b)\in\mathbb{X}$, let $\bar{k}:=w-b>0$. The argument $c^o(z,k,b,\varepsilon):=b\,\mathcal{R}^\star(z;\varepsilon)+k\mathcal{R}(k;\varepsilon)$ will be suppressed when convenient. Suppose $b=0$. Then, $H^{(1)}$ is independent of z and ϑ and the existence of a zero $k\in]0,\bar{k}[$ of $H^{(1)}(z,\cdot;w,0)$ follows from the arguments of Wang (1993) who also shows that (T1) is sufficient for this zero to be unique. Given k, the condition $H^{(2)}(z,k;w,0)=0$ can be solved explicitly for $z>0$ proving the case $b=0$.
Suppose $b>0$. The strategy is to use (7b) to eliminate z reducing (8) to a one-dimensional problem. First, let $\hat{k}\in]0,\bar{k}[$ be arbitrary. We prove existence of a unique $\hat{z}>0$ to satisfy $H^{(2)}(\hat{z},\hat{k};w,b)=0$. Since $\lim_{z\to\infty}c^o(z,k,b,\varepsilon)=\infty$ for each $\varepsilon\in\mathcal{E}$, (U2) implies

$$\lim_{z\to\infty} z\,\vartheta(\varepsilon)\,v'(-)=b^{-1}\lim_{z\to\infty}c^o(z,\hat{k},b,\varepsilon)v'(-)-b^{-1}\hat{k}\mathcal{R}(\hat{k};\varepsilon)\lim_{z\to\infty}v'(-)=\infty.$$

This being true for all $\varepsilon\in\mathcal{E}$ implies $H^{(2)}(z,\hat{k};w,b)<0$ for z sufficiently large. Combined with $H^{(2)}(0,\hat{k};w,b)>0$, this proves existence of $\hat{z}$. Uniqueness follows from (U1) by which

$$\begin{aligned} H^{(2)}_z(z,k;w,b) &= -\mathbb{E}_\nu\left[\vartheta(\cdot)\,v'\big(c^o(z,k,b,\cdot)\big)+b\,z\,\vartheta(\cdot)^2\,v''\big(c^o(z,k,b,\cdot)\big)\right] \\ &< -\mathbb{E}_\nu\left[\vartheta(\cdot)\big(v'\big(c^o(z,k,b,\cdot)\big)+c^o(z,k,b,\cdot)v''\big(c^o(z,k,b,\cdot)\big)\big)\right]\le 0. \end{aligned} \tag{A.1}$$

Let $\hat{Z}(\cdot;w,b):]0,\bar{k}[\longrightarrow\mathbb{R}_{++}$ determine the value $\hat{z}$ for each $\hat{k}\in]0,\bar{k}[$. By (2b) and (T1),

$$H^{(2)}_k(z,k;w,b)=-u''(w-b-k)-\big(1+E_{f'}(k)\big)\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)\,z\,\vartheta(\cdot)v''(-)\right]>0. \tag{A.2}$$

By (A.1), (A.2) and the implicit function theorem, $\hat{Z}(\cdot;w,b)$ is C^1 and strictly increasing since $\hat{Z}_k(k;w,b)=-H^{(2)}_k(\hat{z},k;w,b)/H^{(2)}_z(\hat{z},k;w,b)>0$, for all $k\in]0,\bar{k}[$, $\hat{z}=\hat{Z}(k;w,b)$.
Second, let $\hat{H}^{(1)}(k;w,b):=H^{(1)}(\hat{Z}(k;w,b),k;w,b)$, $k\in]0,\bar{k}[$. We show that $\hat{H}^{(1)}(\cdot;w,b)$ has a unique zero $k'\in]0,\bar{k}[$. Since v' is strictly decreasing, $\mathcal{R}(k;\varepsilon)v'\big(b\,\hat{Z}(k;w,b)\,\vartheta(\varepsilon)+k\mathcal{R}(k;\varepsilon)\big)<\mathcal{R}(k;\varepsilon)v'\big(k\mathcal{R}(k;\varepsilon)\big)$ for all $k\in]0,\bar{k}[$ and $\varepsilon\in\mathcal{E}$. Then, by the Inada conditions

$$\lim_{k\nearrow\bar{k}}\hat{H}^{(1)}(k;w,b)\ge\lim_{k\nearrow\bar{k}}\Big(u'(\bar{k}-k)-\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(k\mathcal{R}(k;\cdot)\big)\right]\Big)=\infty.$$

Let $(k_n)_{n\geq 1}$ be a sequence in $]0,\bar{k}[$ with $\lim_{n\to\infty} k_n = 0$. Since $k \mapsto \hat{Z}(k;w,b)$ and, by (T1), $k \mapsto k\mathcal{R}(k;\varepsilon)$ are increasing, $c_n(\varepsilon) := b\,\hat{Z}(k_n;w,b)\,\vartheta(\varepsilon) + k_n\mathcal{R}(k_n,\varepsilon)$ is bounded from above and $\lim_{n\to\infty}\mathcal{R}(k_n,\varepsilon)\,v'\big(c_n(\varepsilon)\big) = \infty$ for all $\varepsilon \in \mathcal{E}$. Therefore, $\lim_{n\to\infty}\hat{H}^{(1)}(k_n;w,b) = -\infty$. This proves existence of a zero k'. Finally, using (U2) the partial derivatives satisfy

$$
\begin{aligned}
H_z^{(1)}(z,k;w,b) &= -\mathbb{E}_\nu\big[\mathcal{R}(k,\cdot)\,b\,\vartheta(\cdot)\,v''(-)\big] > 0 && \text{(A.3)}\\
H_k^{(1)}(z,k;w,b) &= -u''(-) - \mathbb{E}_\nu\big[\mathcal{R}_k(k;\cdot)\,v'(-) + (1+E_{f'}(k))\mathcal{R}(k;\cdot)^2\,v''(-)\big] > 0. && \text{(A.4)}
\end{aligned}
$$

Combining (A.3) and (A.4) with the monotonicity of $\hat{Z}(\cdot;w,b)$ yields $\hat{H}_k^{(1)}(k;w,b) = H_z^{(1)}(\hat{z},k;w,b)\hat{Z}_k(k;w,b) + H_k^{(1)}(\hat{z},k;w,b) > 0$ for all $k \in]0,\bar{k}[$ and $\hat{z} = \hat{Z}(k;w,b)$. Hence, k' is the unique zero of $\hat{H}^{(1)}(\cdot;w,b)$. Setting $z = \hat{Z}(k';w,b)$ completes the proof. ■

A.2 Proof of Lemma 2.2

(i) The first limit follows from $0 < \mathcal{K}(w,b) < w-b$ for all $x=(w,b)\in\mathbb{X}$. To see the second one, note from (8) that there must be some $\tilde{\varepsilon}\in\mathcal{E}$ for which $\vartheta(\tilde{\varepsilon})\mathcal{Z}(x) \geq \varepsilon_{\min} f'(\mathcal{K}(x))$. Thus, letting $\zeta := \varepsilon_{\min}/\vartheta(\tilde{\varepsilon})$ we have $\mathcal{Z}(x) \geq \zeta f'(\mathcal{K}(x))$ for all $x\in\mathbb{X}$. Combined with the first result and the boundary behavior of f', the claim follows.

(ii)/(iii) We suppress arguments of functions when convenient. Given $x=(w,b)\in\mathbb{X}$, set $z := \mathcal{Z}(x)$, $k := \mathcal{K}(x)$, $\xi = (z,k)$ and write $H = (H^{(1)},H^{(2)})$. Using (A.1), (A.2), (A.3), and (A.4) the Jacobian matrix $D_\xi H$ satisfies $\det D_\xi H = H_z^{(1)}H_k^{(2)} - H_k^{(1)}H_z^{(2)} > 0$. Further, the partial derivatives of H with respect to w and b are given by

$$
\begin{aligned}
H_w^{(1)}(z,k;w,b) &= H_w^{(2)}(z,k;w,b) = u''(w-b-k) < 0 && \text{(A.5)}\\
H_b^{(1)}(z,k;w,b) &= -u''(w-b-k) - \mathbb{E}_\nu\big[\mathcal{R}(k;\cdot)\,\mathcal{R}^\star(z;\cdot)v''\big(-\big)\big] > 0 && \text{(A.6)}\\
H_b^{(2)}(z,k;w,b) &= -u''(w-b-k) - \mathbb{E}_\nu\big[(\mathcal{R}^\star(z;\cdot))^2 v''\big(-\big)\big] > 0. && \text{(A.7)}
\end{aligned}
$$

By the implicit function theorem, using the standard inversion formula for 2×2 matrices

$$
\begin{aligned}
\mathcal{Z}_w(w,b) &= \frac{-H_w^{(1)}[H_k^{(2)} - H_k^{(1)}]}{\det D_\xi H}, & \mathcal{Z}_b(w,b) &= \frac{H_k^{(1)}H_b^{(2)} - H_k^{(2)}H_b^{(1)}}{\det D_\xi H}\\
\mathcal{K}_w(w,b) &= \frac{-H_w^{(1)}[H_z^{(1)} - H_z^{(2)}]}{\det D_\xi H}, & \mathcal{K}_b(w,b) &= \frac{H_z^{(2)}H_b^{(1)} - H_z^{(1)}H_b^{(2)}}{\det D_\xi H}.
\end{aligned}
\tag{A.8}
$$

Since the matrix $D_\xi H(z,k;w,b)$ is non-singular also at any boundary point $(w,0)\in\mathbb{X}_0$, the implicit function theorem implies that the mappings $\mathcal{Z}$ and $\mathcal{K}$ can locally be extended to an open neighborhood around $(w,0)$. Hence, their derivatives are well-defined and continuous also on the boundary $\mathbb{X}_0$ and Lemma 2.2 indeed holds on the entire set $\mathbb{X}$.

(ii) Use $H_z^{(2)} < 0 \leq H_z^{(1)}$ by (A.1), (A.3), and $0 < -H_w^{(1)} < H_b^{(i)}$, $i=1,2$, by (A.5)–(A.7).

(iii) For $\vartheta = \mathrm{id}_\mathcal{E}$ one has $\mathcal{Z}(w,b) = f'(\mathcal{K}(w,b))$ by (8) and (ii) is implied by (i). If, instead, (U3) holds, straightforward calculations give

$$
\begin{aligned}
H_k^{(1)} - H_k^{(2)} &= \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right](1+E_{f'}(k)) - \mathbb{E}_\nu\left[\mathcal{R}_k(k;\cdot)v'(-)\right]\\
H_b^{(1)} - H_b^{(2)} &= \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}^\star(z;\cdot)|v''(-)|\right].
\end{aligned}
$$

By Lemma B.1, $H_k^{(1)} - H_k^{(2)} > 0 \geq H_b^{(1)} - H_b^{(2)}$ which gives $\mathcal{Z}_w < 0 < \mathcal{Z}_b$. Finally,

$$\mathcal{K}_w \mathcal{Z}_b - \mathcal{K}_b \mathcal{Z}_w = \frac{-H_w^{(1)}[H_b^{(2)} - H_b^{(1)}]}{\det D_\xi H} \geq 0. \tag{A.9}$$

■

A.3 Proof of Lemma 3.1

Let $\bar{x}^0 = (\bar{w}^0, 0)$ be the unique fixed point of Φ in $\mathbb{X}_0$ from Assumption 2. First, we show that $\psi(\bar{x}^0) \geq 1$ implies that Φ is expansive. By contradiction, suppose $\psi(\bar{x}^0) \geq 1$ and Φ has a fixed point $\bar{x} = (\bar{w}, \bar{b})$ in $\mathbb{X}_+$. Then, as $\phi_b < 0$ one has $\phi(w, \bar{b}) < \phi(w, 0) \leq w$ for all $w \geq \bar{w}^0$. It follows that $\bar{w} < \bar{w}^0$. Monotonicity of ψ implies $1 \leq \psi(\bar{x}^0) < \psi(\bar{w}^0, \bar{b}) < \psi(\bar{x})$. But this contradicts (11) whose second component clearly implies $\psi(\bar{x}) = 1$.
Second, we show that $\psi(\bar{x}^0) < 1$ implies that Φ has a fixed point $\bar{x} = (\bar{w}, \bar{b}) \in \mathbb{X}_+$. Let $F = (F^{(1)}, F^{(2)}) : \mathbb{X} \longrightarrow \mathbb{R}^2$ be defined by $F^{(1)}(w, b) := w - \phi(w, b)$ and $F^{(2)}(w, b) := \psi(w, b) - 1$. Any value $x \in \mathbb{X}_+$ that satisfies $F(x) = 0$ is a fixed point of Φ.
By uniqueness and stability of $\bar{x}^0$, any $x = (w, b) \in \mathbb{X}_+$ satisfying $w \geq \bar{w}^0$ gives $F^{(1)}(w, b) > w - \phi(w, 0) \geq 0$. Further, let $0 < \underline{w} < \bar{w}^0$ be the unique value for which $\psi(\underline{w}, 0) = 1$ which is well-defined by the monotonicity and boundary properties of ψ. Observe that for any $x = (w, b) \in \mathbb{X}_+$ satisfying $w \leq \underline{w}$, $F^{(2)}(w, b) > \psi(w, 0) - 1 \geq \psi(\underline{w}, 0) - 1 = 0$. Combining both results shows that any fixed point $\bar{x} = (\bar{w}, \bar{b}) \in \mathbb{X}_+$ satisfies $\bar{w} \in \overline{\mathbb{W}} :=]\underline{w}, \bar{w}^0[$.
For any $w \in \overline{\mathbb{W}}$ we have $F^{(1)}(w, 0) < 0$ and $\lim_{b \nearrow w} F^{(1)}(w, b) = w > 0$. Thus, there exists a value $0 < b < w$ such that $F^{(1)}(w, b) = 0$ which is unique by monotonicity of ϕ. Let this value be determined by the implicit function $f^{(1)} : \overline{\mathbb{W}} \longrightarrow \mathbb{R}_{++}$ which is C^1 by the implicit function theorem with derivative $f^{(1)\prime}(w) = (1 - \phi_w(w, b))/\phi_b(w, b)$, where $w \in \overline{\mathbb{W}}$ and $b = f^{(1)}(w)$. Continuity of $F^{(1)}$ implies $\lim_{w \nearrow \bar{w}^0} f^{(1)}(w) = 0$ and $\lim_{w \searrow \underline{w}} f^{(1)}(w) > 0$.
For any $w \in \overline{\mathbb{W}}$ we have $F^{(2)}(w, 0) < 0$ and $\lim_{b \nearrow w} F^{(1)}(w, b) = \infty$. Thus, there exists a value $0 < b < w$ such that $F^{(2)}(w, b) = 0$ which is unique by monotonicity of ψ. Let this value be determined by the implicit function $f^{(2)} : \overline{\mathbb{W}} \longrightarrow \mathbb{R}_{++}$ which is C^1 by the implicit function theorem with derivative $f^{(2)\prime}(w) = -\psi_w(w, b))/\psi_b(w, b) > 0$ where $w \in \overline{\mathbb{W}}$ and $b = f^{(2)}(w)$. Continuity of $F^{(2)}$ implies $\lim_{w \nearrow \bar{w}^0} f^{(2)}(w) > 0$ and $\lim_{w \searrow \underline{w}} f^{(2)}(w) = 0$.
Let $\Delta : \overline{\mathbb{W}} \longrightarrow \mathbb{R}$, $\Delta(w) := f^{(1)}(w) - f^{(2)}(w)$. Any zero $\bar{w} \in \overline{\mathbb{W}}$ of Δ defines a steady state value $\bar{x} = (\bar{w}, f^{(1)}(\bar{w}))$. Existence of such a zero now follows from continuity of Δ and the boundary behavior $\lim_{w \nearrow \bar{w}^0} \Delta(w) < 0$ and $\lim_{w \searrow \underline{w}} \Delta(w) > 0$. For later reference, we also note that the derivative at the steady state is given by

$$\Delta'(\bar{w}) = -\frac{\psi_b(\bar{x}) - \phi_w(\bar{x})\psi_b(\bar{x}) + \psi_w(\bar{x})\phi_b(\bar{x})}{|\phi_b(\bar{x})\psi_b(\bar{x})|}. \tag{A.10}$$

By the boundary behavior of Δ, there is always a steady state at which $\Delta'(\bar{w}) \leq 0$. ■

A.4 Proof of Lemma 3.2

By contradiction, suppose there exists $x_0 = (w_0, b_0) \in \mathbb{X}_+$ such that $x_t := \Phi^t(x_0) \in \mathbb{X}$ for all $t \geq 0$. Let $x_0^0 := (w_0, 0)$ and $x_t^0 := \Phi^t(x_0^0) \in \mathbb{X}$ for all $t \geq 0$. Clearly, $x_t \in \mathbb{X}_+$ and $x_t^0 \in \mathbb{X}_0$ for all $t \geq 0$. Stability of $\bar{x}^0$ due to Assumption 2 implies $\lim_{t\to\infty} x_t^0 = \bar{x}^0 = (\bar{w}^0, 0)$.

Further, Φ being expansive implies $\psi(\bar{x}^0) > 1$ by Lemma 3.1 as $\psi(\bar{x}^0) = 1$ is excluded by assumption. A simple induction argument using the monotonicity properties of Φ shows that $w_t^0 > w_t > b_t > 0$ for all t. Further, the induced sequences $\psi_t^0 := \psi(x_t^0)$ and $\psi_t := \psi(x_t)$, $t \geq 0$ satisfy $\psi_t > \psi_t^0$ for all $t \geq 0$ and $\lim_{\to\infty} \psi_t^0 = \psi(\bar{x}^0) > 1$ by continuity of ψ and stability of $\bar{x}^0$. Thus, there exists $T \geq 0$ such that $\psi_t > \psi_t^0 > 1$ for all $t \geq T$ and the sequence $(b_t)_{t \geq T}$ is strictly increasing, i.e., $b_{t+1} = \psi_t b_t > b_t$ for all $t \geq T$. As $b_t < w_t < w_t^0$ for all $t \geq 0$, $\bar{b} := \lim_{t\to\infty} b_t$ exists and satisfies $b_T < \bar{b} < \bar{w}^0$. But then, $1 = \lim_{t\to\infty} \frac{b_{t+1}}{b_t} = \lim_{t\to\infty} \psi_t$ which contradicts $\lim_{t\to\infty} \psi_t \geq \lim_{t\to\infty} \psi_t^0 = \psi(\bar{x}^0) > 1$. ■

A.5 Proof of Lemma 3.3

At any steady state $\bar{x} = (\bar{w}, \bar{b}) \in \mathbb{X}_+$ the trace and determinant of the Jacobian $D\Phi(\bar{x})$ read $\mathrm{tr} D\Phi(\bar{x}) = 1 + \phi_w(\bar{x}) + \bar{b}\,\psi_b(\bar{x})$ and $\det D\Phi(\bar{x}) = \phi_w(\bar{x}) + \bar{b}[\phi_w(\bar{x})\psi_b(\bar{x}) - \phi_b(\bar{x})\psi_w(\bar{x})]$. By the properties of ϕ and ψ, $\mathrm{tr} D\Phi(\bar{x}) > 1$, $\det D\Phi(\bar{x}) > 0$ and $\mathrm{tr} D\Phi(\bar{x}) = 1 + \det D\Phi(\bar{x}) - \zeta\Delta'(\bar{w})$ where $\zeta > 0$ and Δ is defined as in the proof of Lemma 3.3. By uniqueness of the steady state, $\Delta'(\bar{w}) < 0$ as $\Delta'(\bar{w}) = 0$ would imply a non-hyperbolic steady state. Hence, $\mathrm{tr} D\Phi(\bar{x}) > 1 + \det D\Phi(\bar{x})$ implying saddle-path stability of $\bar{x}$, cf. Galor (2007, p.88). ■

A.6 Proof of Lemma 3.4

Define $\overline{\mathbb{X}}$ as in (13). Note that $\bar{x} \in \overline{\mathbb{X}}$ and that $\mathbb{M} \subset \overline{\mathbb{X}}$.

(i) *Step 1:* $\mathbb{M}$ is a one-dimensional C^1-manifold. By the *Stable Manifold Theorem* (cf. Nitecki (1971)), there is an open neighborhood $\mathbb{U} \subset \mathbb{X}_+ \cap \mathbb{Y}$ of $\bar{x}$ such that the locally stable set $\mathbb{M}^{\mathrm{loc}} := \{x \in \mathbb{X}_+ \,|\Phi^n(x) \in \mathbb{U}\ \forall n \geq 1 \wedge \lim_{n\to\infty} \Phi^n(x) = \bar{x}\}$ is a one-dimensional manifold which is as smooth as Φ, i.e., C^1. By Nitecki (1971, p.89) or Galor (1992, p.1371, Definition 4), the globally stable manifold defined in (12) obtains as $\mathbb{M} = \cup_{n\geq 0}\Phi^{-n}(\mathbb{M}^{\mathrm{loc}})$. Exploiting Assumption 4, $\mathbb{M}$ inherits the smoothness of $\mathbb{M}^{\mathrm{loc}}$ and is thus a one-dimensional C^1-manifold. The same arguments are used in Galor (1992, p.1371, Corollary 3).

Step 2: $\mathbb{M}$ is the graph of a strictly increasing function $\mathcal{M} : \mathbb{W} \longrightarrow \mathbb{R}_{++}$, $\mathbb{W} \subset \mathbb{R}_{++}$. By Lemma B.2, for each $\tilde{w} > 0$ there exists at most one $0 < \tilde{b} < \tilde{w}$ such that $(\tilde{w}, \tilde{b}) \in \mathbb{M}$. Let $\mathbb{W}$ be the set of all $\tilde{w} > 0$ for which such a value $\tilde{b}$ exists. Then, $\bar{w} \in \mathbb{W}$ and $\mathbb{M}$ is the graph of $\mathcal{M} : \mathbb{W} \longrightarrow \mathbb{R}_{++}$ defined via $\mathcal{M}(\tilde{w}) := \tilde{b}$. Lemma B.2 also implies that $\mathcal{M}$ is increasing.

Step 3: $\mathbb{W}$ is an interval and $\mathcal{M}$ is continuous. As $\mathbb{M}$ is C^1, there exists an open neighborhood $\mathbb{V} \subset \mathbb{M}$ of $\bar{x}$, an open subset $\mathbb{U} \subset \mathbb{R}$ and a C^1-diffeomorphism $\varphi : \mathbb{V} \longrightarrow \mathbb{U}$. W.l.o.g., let $\mathbb{U}$ be an interval and $\mathbb{V} \subset \mathbb{M}^{\mathrm{loc}}$ (otherwise, choose an open interval $\tilde{\mathbb{U}} \subset \mathbb{U}$ containing $\varphi(\bar{x})$ small enough such that $\varphi^{-1}(\tilde{\mathbb{U}}) \subset \mathbb{M}^{\mathrm{loc}}$ and switch to $\tilde{\mathbb{V}} := \varphi^{-1}(\tilde{\mathbb{U}})$ and $\tilde{\varphi} := \varphi_{|\tilde{\mathbb{V}}}$). By Dugundji (1970, p.108, Theorem I.4), $\mathbb{V} = \varphi^{-1}(\mathbb{U})$ being the image of an open and connected set under a homeomorphism is an open and connected subset of $\mathbb{M}$ containing $\bar{x}$. Let $x \in \mathbb{M}$ be arbitrary. By (12), $\lim_{n\to\infty} \Phi^n(x) = \bar{x}$ implying $\Phi^n(x) \in \mathbb{V}$ for n large enough, i.e., $x \in \Phi^{-n}(\mathbb{V})$. Since x was arbitrary and $\mathbb{V} \subset \mathbb{M}^{\mathrm{loc}}$, $\mathbb{M} = \cup_{n\geq 0}\Phi^{-n}(\mathbb{V})$. Continuity of Φ^{-n} and Theorem I.4 in Dugundji (1970) imply that each $\Phi^{-n}(\mathbb{V})$ is a connected set containing $\bar{x}$. By (12) and Theorem I.5 in Dugundji (1970, p.108), $\mathbb{M}$ is connected and so are $\mathbb{W}$ and $\mathbb{B} := \mathcal{M}(\mathbb{W})$ as the images of $\mathbb{M}$ under the continuous projections $\pi_1 : (w, b) \mapsto w$ and $\pi_2 : (w, b) \mapsto b$. Thus, both $\mathbb{W}$ and $\mathbb{B}$ are intervals. Suppose $\mathcal{M}$ were not continuous at some interior point $w_0 \in \mathbb{W}$. Then, there exists $\varepsilon > 0$ such that for all $\delta > 0$ sufficiently small there is some $\tilde{w} \in]w_0 - \delta, w_0 + \delta[$ for which $|\mathcal{M}(\tilde{w}) - \mathcal{M}(w_0)| \geq \varepsilon$. Then, by strict monotonicity of $\mathcal{M}$, for all $\delta > 0$, either $\mathcal{M}(w_0) \geq \varepsilon + \mathcal{M}(w_0 - \delta)$ or $\mathcal{M}(w_0 + \delta) \geq \varepsilon + \mathcal{M}(w_0)$. In particular, there is no $w \in \mathbb{W}$ for which $\mathcal{M}(w) \in [\mathcal{M}(w_0) - \frac{2}{3}\varepsilon, \mathcal{M}(w_0) - \frac{1}{3}\varepsilon]$. Conclude that $\mathbb{B} \subset]0, \mathcal{M}(w_0) - \frac{2}{3}\varepsilon[\cup]\mathcal{M}(w_0) - \frac{1}{3}\varepsilon, \infty[$, i.e., $\mathbb{B}$ is separated which is a contradiction.

Step 4: $\mathcal{M}$ is C^1. Let w_0 be an interior point of $\mathbb{W}$. Since $\mathbb{M}$ is C^1, there exist an open neighborhood $\mathbb{V}_0 \subset \mathbb{M}$ of $x_0 := (w_0, \mathcal{M}(w_0))$, an open set $\mathbb{U}_0 \subset \mathbb{R}$ and a C^1-diffeomorphism $\Gamma = (\Gamma_1, \Gamma_2) : \mathbb{U}_0 \longrightarrow \mathbb{V}_0$. Let $F := (\mathrm{id}_{\mathbb{W}}, \mathcal{M}) : \mathbb{W} \longrightarrow \mathbb{M}$ which is continuous by Step 3 and so is the inverse $F^{-1} = \pi_1$ which is the projection defined above. Define $\mathbb{W}_0 := \pi_1(\mathbb{V}_0)$ which is open since π_1 is open. Thus, $\Gamma_1 = F^{-1} \circ \Gamma : \mathbb{U}_0 \longrightarrow \mathbb{W}_0$ is C^1 and the inverse $\Gamma_1^{-1} = \Gamma^{-1} \circ F : \mathbb{W}_0 \longrightarrow \mathbb{U}_0$ is at least continuous. The strategy is to show that Γ_1^{-1} is even C^1. Suppose $\Gamma_1'(\tilde{u}) = 0$ for some $\tilde{u} \in \mathbb{U}_0$. Let $\tilde{w} := \Gamma_1(\tilde{u})$. Since $\Gamma_2 = \mathcal{M} \circ \Gamma_1$ and $\frac{\mathcal{M}(w) - \mathcal{M}(\tilde{w})}{w - \tilde{w}}$ takes values in the unit interval[10] for all $w > 0$, $\Gamma_2'(\tilde{u}) = \Gamma_1'(\tilde{u}) \lim_{w \to \tilde{w}} (\mathcal{M}(w) - \mathcal{M}(\tilde{w}))/(w - \tilde{w}) = 0$. Adopting an argument from Villanacci et al.(2002, p.39), let Ψ be a C^1-extension of Γ^{-1} to an open set in $\mathbb{R}^2$ containing $\mathbb{V}_0$, i.e., $\Psi_{|\mathbb{V}_0} = \Gamma^{-1}$. Then, $(\Psi \circ \Gamma)'(\tilde{u}) = \partial_1 \Psi(\Gamma(\tilde{u}))\Gamma_1'(\tilde{u}) + \partial_2 \Psi(\Gamma(\tilde{u}))\Gamma_2'(\tilde{u}) = 0$ which contradicts $(\Psi \circ \Gamma)_{|\mathbb{U}_0} = \mathrm{id}_{\mathbb{U}_0}$ implying $(\Psi \circ \Gamma)'(\tilde{u}) = 1$. Conclude $\Gamma_1'(u) \neq 0$ for all $u \in \mathbb{U}_0$. Then, by the inverse function theorem $(\Gamma_1^{-1})'(w) = 1/\Gamma_1'(\Gamma_1^{-1}(w))$ for all $w \in \mathbb{W}_0$. Since Γ_1 is C^1 and Γ_1^{-1} continuous, $(\Gamma_1^{-1})'$ is well-defined and continuous. Thus, Γ_1 is a C^1-diffeomorphism and so is $F = \Gamma \circ \Gamma_1^{-1}$ restricted to $\mathbb{W}_0$. Hence, $\mathcal{M}$ is C^1 on $\mathbb{W}_0$ and, in particular, at w_0.

Step 5: $\phi_{\mathbb{M}}(w) := \phi(w, \mathcal{M}(w))$, $w \in \mathbb{W}$ is increasing. We first show that $\phi_{\mathbb{M}}$ is non-decreasing, i.e., $\mathcal{M}' \leq -\phi_w/\phi_b < 1$. By contradiction, suppose $\mathcal{M}'(\tilde{w}) > -\phi_w(\tilde{w}, \tilde{b})/\phi_b(\tilde{w}, \tilde{b})$ for some interior point $\tilde{w} \in \mathbb{W}$ where $\tilde{b} := \mathcal{M}(\tilde{w})$. Then, $\mathcal{M}'(\tilde{w}) > -\psi_1(\tilde{w}, \tilde{b})/\psi_2(\tilde{w}, \tilde{b})$. Let $\psi_{\mathbb{M}}(w) := \psi(w, \mathcal{M}(w))$, $w \in \mathbb{W}$. By continuity, $\phi_{\mathbb{M}}$ is locally strictly decreasing while $\psi_{\mathbb{M}}$ is locally strictly increasing around $\tilde{w}$. Let $\hat{w} > \tilde{w}$ be close to $\tilde{w}$ and $\hat{b} := \mathcal{M}(\hat{w})$. Then, $(\hat{w}, \hat{b}), (\tilde{w}, \tilde{b}) \in \mathbb{M}$ and $\hat{w}_1 := \phi_{\mathbb{M}}(\hat{w}) < \phi_{\mathbb{M}}(\tilde{w}) =: \tilde{w}_1$ while $\hat{b}_1 := \hat{b}\, \psi_{\mathbb{M}}(\hat{w}) > \tilde{b}\, \psi_{\mathbb{M}}(\tilde{w}) =: \tilde{b}_1$. But $\mathbb{M}$ being self-supporting under Φ implies $(\tilde{w}_1, \tilde{b}_1) = \Phi(\tilde{w}, \tilde{b}) \in \mathbb{M}$ and $(\hat{w}_1, \hat{b}_1) = \Phi(\hat{w}, \hat{b}) \in \mathbb{M}$, i.e., $\tilde{b}_1 = \mathcal{M}(\tilde{w}_1)$ and $\hat{b}_1 = \mathcal{M}(\hat{w}_1)$ which contradicts that $\mathcal{M}$ is strictly increasing. To see that $\phi_{\mathbb{M}}$ is even strictly increasing, suppose $\phi_{\mathbb{M}}(\hat{w}) = \phi_{\mathbb{M}}(\tilde{w})$ for some $\hat{w} > \tilde{w}$. Then, $\phi_{\mathbb{M}}$ must be constant on the interval $[\tilde{w}, \hat{w}]$ while $\psi_{\mathbb{M}}$ is weakly increasing. Repeating the previous argument, $\hat{w}_1 = \tilde{w}_1$ and $\hat{b}_1 > \tilde{b}_1$ leading to the same contradiction.

Step 6: $\mathbb{W} = \mathbb{R}_{++}$. By Step 5, $\phi_{\mathbb{M}}^{-1} : \mathbb{W}^\star \longrightarrow \mathbb{W}$ is well-defined where $\mathbb{W}^\star := \phi_{\mathbb{M}}(\mathbb{W})$ is an interval with the same structure (left-open/closed and right-open/closed) as $\mathbb{W}$. By (12), $\phi_{\mathbb{M}}$ has $\bar{w}$ as its unique fixed point which is globally asymptotically stable on $\mathbb{W}$. Therefore,

$$\forall w \in \mathbb{W} : \ \phi_{\mathbb{M}}(w) \gtreqless w \ \Leftrightarrow \ w \lesseqgtr \bar{w} \quad \text{and} \quad \forall w \in \mathbb{W}^\star : \ \phi_{\mathbb{M}}^{-1}(w) \lesseqgtr w \ \Leftrightarrow \ w \lesseqgtr \bar{w}. \qquad \text{(A.11)}$$

Define $w_{\inf} := \inf \mathbb{W} < \bar{w} < \sup \mathbb{W} =: w_{\sup}$ and $w^\star_{\inf} := \inf \mathbb{W}^\star < \bar{w} < \sup \mathbb{W}^\star =: w^\star_{\sup}$. By (12) and Assumption 4, Φ is a homeomorphism between $\mathbb{M}$ and $\mathbb{M} \cap \mathbb{Y}$ from which we infer that $\mathbb{W}^\star = \mathbb{W} \cap]0, w_{\max}[$ and , therefore, $w^\star_{\inf} = w_{\inf}$ and $w^\star_{\sup} = \min\{w_{\sup}, w_{\max}\}$.

We show $w_{\inf} = 0$. Choose $w_0 \in \mathbb{W}$ such that $w_{\inf} < w_0 < \bar{w}$. For $n \geq 0$, let $w_{n+1} = \phi_{\mathbb{M}}^{-1}(w_n)$ and $b_n := \mathcal{M}(w_n)$ which are well-defined as $\phi_{\mathbb{M}}^{-1}$ maps $]w_{\inf}, \bar{w}[$ into itself. Also note that $x_n := (w_n, b_n) \in \mathbb{M}$ and $x_n = \Phi^{-1}(x_{n-1})$ for all $n \geq 1$. By (A.11), $(w_n)_{n \geq 1}$ is strictly decreasing and converges to some value $w_\infty \geq w_{\inf}$. Suppose $w_\infty > 0$. By monotonicity of $\mathcal{M}$, $(b_n)_{n \geq 1}$ is strictly decreasing and converges to $b_\infty \leq w_\infty$. Suppose $w_\infty = b_\infty$. Then,

[10] This follows from monotonicity of $\mathcal{M}$ and a straightforward modification of the contradiction argument employed in Step 5 below where $\mathcal{M}'(\tilde{w})$ needs to be replaced by the difference quotient $\frac{\Delta b}{\Delta w} := \frac{\mathcal{M}(w) - \mathcal{M}(\tilde{w})}{w - \tilde{w}}$.

$\lim_{n\to\infty} \psi(w_n, b_n) = \infty$ by the properties of ψ and, since $b_\infty > 0$, $(w_n, b_n) \notin \overline{\mathbb{X}} \supset \mathbb{M}$ for large n, which is a contradiction. Conclude that $\lim_{n\to\infty} x_n = x_\infty := (w_\infty, b_\infty) \in \mathbb{X}$. As $\Phi(x_{n+1}) = x_n$ for all n, continuity of Φ gives $\lim_{n\to\infty} \Phi(x_n) = x_\infty = \Phi(x_\infty)$. Thus, x_∞ is a fixed point of Φ satisfying $0 < w_\infty < \bar{w} < \bar{w}^0$, which contradicts either Assumption 2 or 3. Conclude that $w_\infty = 0$ which implies $w_{\inf} = 0$.
We show $w_{\sup} = \infty$. Suppose $w_{\sup} < w_{\max}$. Then $w^\star_{\sup} = w_{\sup} < \infty$ and, by (A.11) $\phi_{\mathbb{M}}$ maps $]\bar{w}, w_{\sup}[$ into itself. One can now choose $w_0 \in]\bar{w}, w_{\sup}[$ and modify the arguments from the previous paragraph to obtain a contradiction. Conclude that $w_{\sup} \geq w_{\max} = w^\star_{\sup}$. Let $(w_n)_{n\geq 1}$ be a strictly increasing sequence in $\mathbb{W}$ converging to $w_{\sup}$. Then, $(\phi_{\mathbb{M}}(w_n))_{n\geq 0}$ converges to $w_{\max}$. But, by definition of $w_{\max}$, this is only possible if $w_{\sup} = \infty$.

(ii) Claim (a) follows from Lemma B.2 and Assumptions 2 and 3 while (b) follows from (12), (i), and (A.11). To show (c), assume by contradiction that $b > \mathcal{M}(w)$ but $x = (w, b) \in \overline{\mathbb{X}}$. Define $x_t = (w_t, b_t) := \Phi^t(x)$ and $\hat{x}_t = (\hat{w}_t, \hat{b}_t) := \Phi^t(\hat{x})$ where $\hat{x} := (w, \mathcal{M}(w))$. Note that $\hat{x}_t \in \mathbb{M}$ for all $t \geq 0$ and $\lim_{t\to\infty} \hat{x}_t = (\bar{w}, \bar{b})$. Using Assumption 1, an induction argument yields $0 < \hat{b}_t < b_t < w_t < \hat{w}_t$ for all t. Define $\beta_t := b_t/\hat{b}_t$ to observe that $\beta_0 > 1$ and $\beta_{t+1} = \beta_t \psi(x_t)/\psi(\hat{x}_t) > \beta_t$ for all $t \geq 0$. Hence, $\lim_{t\to\infty} \beta_t = \bar{\beta} > 1$ and $\lim_{t\to\infty} b_t = \bar{\beta}\bar{b} =: \bar{b}' > \bar{b}$ exist. Since w_t remains bounded, $x_t \in \mathbb{X}$ for all t only if $\bar{b}' < \infty$ which requires $\lim_{t\to\infty} \psi(x_t) = 1$ by (11). But, by the previous properties $\lim_{t\to\infty} \psi(x_t) \geq \lim_{t\to\infty} \psi(\hat{w}_t, b_t) = \psi(\bar{w}, \bar{b}') > \psi(\bar{w}, \bar{b}) = 1$ which is a contradiction. ■

A.7 Proof of Lemma 4.2

Let $\varepsilon \in \mathcal{E}$ be given and define $\phi^0(w; \varepsilon) := \mathcal{W}(\mathcal{K}^0(w); \varepsilon)$ for $w > 0$ where $k = \mathcal{K}^0(w)$ is the unique solution to $u'(w - k) = \mathbb{E}_\nu[\mathcal{R}(k; \cdot)v'(k\mathcal{R}(k; \cdot))]$. Any steady state of $\Phi(\cdot; \varepsilon)$ in $\mathbb{X}_0$ is of the form $\bar{x}^0 = (\bar{w}^0, 0)$ where $\bar{w}^0 > 0$ is a fixed point of $\phi^0(\cdot; \varepsilon)$. We show that any such steady state satisfies $\phi^0_w(\bar{w}^0; \varepsilon) < 1$. For any $w > 0$ and $k = \mathcal{K}^0(w)$, the derivative reads

$$\phi^0_w(w; \varepsilon) = \frac{E_f(k)}{1 - E_f(k)} \frac{\phi^0(w; \varepsilon)}{w} \frac{w\mathcal{K}_0'(w)}{k} E_{f'}(k). \tag{A.12}$$

By (T2), the first factor in (A.12) is positive but strictly less than one. The second one equals unity at any steady state. Finally, note that the derivative of $\mathcal{K}_0$ satisfies

$$\begin{aligned} 0 < \mathcal{K}_0'(w) &= \frac{1}{1 + E_{f'}(k)\frac{u'(w-k)}{k|u''(w-k)|} + (1 - E_{f'}(k))\frac{\mathbb{E}_\nu[k\mathcal{R}(k;\cdot)^2|v''(k\mathcal{R}(k;\cdot))|]}{k|u''(w-k)|}} \\ &\overset{(T1)}{\leq} \frac{1}{1 + E_{f'}(k)\frac{u'(w-k)}{k|u''(w-k)|}} \overset{(U4)}{\leq} \frac{k}{k + E_{f'}(k)(w-k)} \overset{(T1)}{\leq} \frac{1}{E_{f'}(k)} \frac{k}{w}. \end{aligned} \tag{A.13}$$

Thus, the last factor in (A.12) is also bounded by unity, as was to be shown.
If the additional conditions hold, then $\phi(w; \varepsilon) > w$ for w small while (U5) ensures that $\lim_{w\to\infty} \mathcal{K}_0(w) = \infty$. This and the boundary behavior of f implies $\phi(w; \varepsilon) < \varepsilon f(\mathcal{K}_0(w)) < \mathcal{K}_0(w) < w$ for w sufficiently large and yields the existence of a non-trivial steady state. ■

A.8 Proof of Lemma 4.3

We show that if $\Phi = \Phi(\cdot; \varepsilon)$ has a steady state in $\mathbb{X}_+$, it will be unique. Defining Δ as in the proof of Lemma 3.3, it suffices to show that $\Delta'(\bar{w}) < 0$ at any steady state $\bar{x} = (\bar{w}, \bar{b}) \in \mathbb{X}_+$.

For brevity, let $\bar{k} := \mathcal{K}(\bar{x})$ and $\bar{z} := \mathcal{Z}(\bar{x})$. As the denominator in (A.10) is positive, one verifies directly that $\Delta'(\bar{w}) < 0$ if and only if

$$\mathcal{Z}_b(\bar{x}) - E_{f'}(\bar{k})\mathcal{R}(\bar{k};\varepsilon)[\mathcal{K}_w(\bar{x})\mathcal{Z}_b(\bar{x}) - \mathcal{K}_b(\bar{x})\mathcal{Z}_w(\bar{x})] > 0. \tag{A.14}$$

If $\vartheta = \mathrm{id}_{\mathcal{E}}$, the bracketed term in (A.14) is zero and the claim follows from Lemma 2.2 (iii). If $\vartheta \neq \mathrm{id}_{\mathcal{E}}$, use (A.8) and (A.9) to observe that (A.14) is positive, iff $M > 0$ where

$$M := H_k^{(1)}H_b^{(2)} - H_k^{(2)}H_b^{(1)} + E_{f'}(\bar{k})\mathcal{R}(\bar{k};\varepsilon)H_w^{(1)}(H_b^{(2)} - H_b^{(1)}).$$

Let $M_1 := \mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)\,|v'(-)|]$, $M_2 := \mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)^2\,|v''(-)|]$, $M_3 := \mathbb{E}_\nu[(\mathcal{R}^\star(\bar{z};\cdot))^2\,|v''(-)|]$ and $M_4 := \mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)\,\mathcal{R}^\star(\bar{z};\cdot)\,|v''(-)|]$. Using the functional forms (A.1)–(A.4), and (A.5)–(A.7), tedious but straightforward calculations reveal that $M = A + B + C$ where

$$\begin{aligned} A &:= |u''(-)|\Big[-\tfrac{f''(\bar{k})}{f'(\bar{k})}M_1 + m(M_3 - M_4) + (1 + E_{f'}(\bar{k}))(M_2 - M_4)\Big] \\ m &:= 1 - E_{f'}(\bar{k})\mathcal{R}(\bar{k};\varepsilon),\ B := -\tfrac{f''(\bar{k})}{f'(\bar{k})}M_1M_3,\ C := (1 + E_{f'}(\bar{k}))\Big[M_2M_3 - \big(M_4\big)^2\Big]. \end{aligned}$$

By Lemma B.1(b), $M_2 \geq M_4$ and $M_3 \geq M_4$ which implies $C \geq 0$ by (T1). Also, $B > 0$. Suppose $m \geq 0$. Then, $A > 0$ by (T1) which implies $M > 0$. Conversely, suppose $-mM_4 > 0$. By (8) and (U4), $M_1 = u'(\bar{w} - \bar{b} - \bar{k}) \geq (\bar{w} - \bar{b} - \bar{k})|u''(\bar{w} - \bar{b} - \bar{k})|$ which implies $B \geq -f''(\bar{k})/f'(\bar{k})(\bar{w} - \bar{b} - \bar{k})|u''(-)|M_3$. By (T1), $(1 + E_{f'}(\bar{k}))(M_2 - M_4) \geq 0$. By (U3), $M_1 = \theta^{-1}(\bar{k}M_2 + \bar{b}M_3)$ implying $M_1 > \bar{b}M_3$ by (U1). Combining the four inequalities gives

$$A + B > |u''(-)|M_3\Big[(1 + E_{f'}(\bar{k})) - \frac{f''(\bar{k})}{f'(\bar{k})}(\bar{w} - \bar{k}\mathcal{R}(\bar{k};\varepsilon))\Big].$$

Both terms in brackets are non-negative due to (T1) and (T2). Hence, $M > 0$. ■

A.9 Proof of Lemma 4.4

Given $\varepsilon \in \mathcal{E}$, let $x' = (w', b') \in \mathbb{Y}_\varepsilon$ arbitrary. We determine a unique $x = (w, b) \in \mathbb{X}_+$ such that $\Phi(x;\varepsilon) = x'$. As $w' \in]0, w_\varepsilon^{\max}[$, there is a unique $k' > 0$ such that $w' = \mathcal{W}(k';\varepsilon)$. The value z' then follows from the first order conditions $\mathbb{E}_\nu[\mathcal{R}^\star(z';\cdot)v'(b'\vartheta(\cdot)/\vartheta(\varepsilon) + k'\mathcal{R}(k';\cdot))] = \mathbb{E}_\nu[\mathcal{R}(k';\cdot)v'(b'\vartheta(\cdot)/\vartheta(\varepsilon) + k'\mathcal{R}(k';\cdot))]$ from which $b = b'/(z'\vartheta(\varepsilon))$ can be inferred. Finally, w is the unique solution to $u'(w - b - k') = \mathbb{E}_\nu[\mathcal{R}(k';\cdot)v'(b\mathcal{R}^\star(z';\cdot) + k'\mathcal{R}(k';\cdot))]$ which is well-defined due to the Inada conditions and (U5) and ensures that $k' = \mathcal{K}(x)$ and $z' = \mathcal{Z}(x)$. Hence, Φ^{-1} is well-defined. As Φ is C^1 by Lemma 2.2 and $\det D\Phi(x) > 0$, $D\Phi^{-1}(x') = [D\Phi(x)]^{-1}$ is continuous by the inverse function theorem. ■

A.10 Proof of Lemma 4.5

By contradiction, suppose $\varepsilon < \varepsilon'$ but $b_0 := \mathcal{M}_\varepsilon(w_0) \geq \mathcal{M}_{\varepsilon'}(w_0) =: b_0'$ for some $w_0 > 0$. Let $x_0 := (w_0, b_0)$ and $x_0' := (w_0, b_0')$. Using (10a,b) and an induction argument in conjunction with Lemma 2.2 and the multiplicative structure of shocks, the sequences $\{x_t\}_{t\geq 0}$ and $\{x_t'\}_{t\geq 0}$ defined as $x_t = (w_t, b_t) := \Phi(x_{t-1};\varepsilon)$ and $x_t' = (w_t', b_t') := \Phi(x_{t-1}';\varepsilon')$ satisfy $w_t < w_t'$ and $b_t \geq b_t'$ for all $t > 0$. Therefore, $\bar{x}_\varepsilon = (\bar{w}_\varepsilon, \bar{b}_\varepsilon) := \lim_{t\to\infty} x_t$ and

$\bar{x}_{\varepsilon'} = (\bar{w}_{\varepsilon'}, \bar{b}_{\varepsilon'}) := \lim_{t\to\infty} x'_t$ satisfy $\bar{w}_\varepsilon \le \bar{w}_{\varepsilon'}$ and $\bar{b}_\varepsilon \ge \bar{b}_{\varepsilon'}$. By Lemma 2.2 (iii), however, the steady state property $\mathcal{Z}(\bar{x}_\varepsilon) = \mathcal{Z}(\bar{x}_{\varepsilon'}) = \frac{1}{\bar{\vartheta}}$ requires $\bar{x}_\varepsilon = \bar{x}_{\varepsilon'}$ implying $\mathcal{K}(\bar{x}_\varepsilon) = \mathcal{K}(\bar{x}_{\varepsilon'}) =: \bar{k}$. But this contradicts $\bar{w}_\varepsilon = \mathcal{W}(\bar{k}, \varepsilon) < \mathcal{W}(\bar{k}, \varepsilon') = \bar{w}_{\varepsilon'}$. Conclude that $\mathcal{M}^{\text{crit}} = \mathcal{M}_{\varepsilon_{\min}}$ in (16). Using this, $\vartheta \equiv \bar{\vartheta}$, and the properties of Φ and $\mathbb{M}_{\varepsilon_{\min}}$, $b \le \mathcal{M}^{\text{crit}}(w)$ implies $\Phi^{(2)}(w, b; \varepsilon) = \Phi^{(2)}(w, b; \varepsilon_{\min}) \le \Phi^{(2)}(w, \mathcal{M}^{\text{crit}}(w); \varepsilon_{\min}) = \mathcal{M}^{\text{crit}}(\Phi^{(1)}(w, \mathcal{M}^{\text{crit}}(w); \varepsilon_{\min})) \le \mathcal{M}^{\text{crit}}(\Phi^{(1)}(w, b; \varepsilon_{\min})) \le \mathcal{M}^{\text{crit}}(\Phi^{(1)}(w, b; \varepsilon))\ \forall \varepsilon \in \mathcal{E}$. Thus, condition (17) holds. ■

A.11 Proof of Lemma 4.6

(i) The unique bubbly steady state can be obtained by direct computations and its stability properties follow from the same arguments used in the proof of Lemma 3.3.
(ii) Let $\beta_t := b_t/w_t$ for $t \ge 0$. Using (20a,b) gives $\beta_{t+1} = \phi(\beta_t) := \frac{\alpha}{1-\alpha}[\gamma - \beta_t]^{-1}\beta_t$, $t \ge 0$. The map ϕ has $\bar{\beta}$ as its unique non-trivial fixed point which is unstable. Moreover, $\beta_0 < \bar{\beta}$ implies $\lim_{t\to\infty} \beta_t = 0$ and $\beta_0 > \bar{\beta}$ implies that $\phi^{t_0}(\beta_0) > \gamma$ for some finite t_0. Hence, $b_0 = \bar{\beta} w_0$ is necessary for $(w_0, b_0) \in \mathbb{M}_\varepsilon$ and each such initial state converges to $\bar{x}$. ■

A.12 Proof of Lemma 5.1

(i) Define β_t as in the previous proof. By (29a,b), $\beta_{t+1} = \phi(\beta_t) := \frac{\alpha}{1-\alpha}\frac{1}{\gamma} L(\beta_t)/\kappa(L(\beta_t))\beta_t$, $t \ge 0$. Using the properties of L and κ, ϕ has $\bar{\beta} > 0$ as its unique non-trivial steady state. As any bubbly steady state of (29a, b) must satisfy $\bar{b}_\varepsilon = \bar{\beta}\bar{w}_\varepsilon$, one obtains $\bar{w}_\varepsilon$ as the unique solution to $w = \Phi^{(1)}(w, \bar{\beta} w; \varepsilon)$. The stability properties follow from the same arguments used in the proof of Lemma 3.3.
(ii) Noting that the steady state in (i) satisfies $\phi'(\bar{\beta}) > 1$ and is, therefore, unstable, an analogous reasoning as in the proof of Lemma 4.6(ii) yields the claim. ■

B Auxiliary results

Lemma B.1
In addition to (T1), (U1), and (U2), let (U3) hold. Then, for all $(w, b) \in \mathbb{X}$, the solutions $z := \mathcal{Z}(w, b)$ and $k := \mathcal{K}(w, b)$ to (8) satisfy the following inequalities:

(a) $k\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right] = -b\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}^\star(z;\cdot)|v''(-)|\right]$.

(b) $\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right] \ge 0 \ge \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}^\star(z;\cdot)|v''(-)|\right]$.

Proof of Lemma B.1.
(a) By (8), $0 = H^{(1)}(z, k; w, b) - H^{(2)}(z, k; w, b) = \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))v'(-)\right]$. By (U3), $v'(c) = \theta^{-1}c|v''(c)|$ for all $c = b\mathcal{R}^\star(z;\cdot) + k\mathcal{R}(k,\varepsilon) > 0$ which yields (a).
(b) As $\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right] \ge \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - \mathcal{R}^\star(z;\cdot))\mathcal{R}^\star(z;\cdot)|v''(-)|\right]$ and, by (a), the two sides are either both zero or have opposite signs, the claim follows. ■

Lemma B.2
Define Φ as in (11) and let $\hat{x} \neq \tilde{x}$ be distinct points in $\overline{\mathbb{X}}$ such that $\hat{w} \ge \tilde{w}$ and $\hat{b} \le \tilde{b}$. Suppose $\hat{x}_n := \Phi^n(\hat{x})$, $n \ge 0$ and $\tilde{x}_n := \Phi^n(\tilde{x})$, $n \ge 0$ converge to $\hat{x}^* = (\hat{w}^*, \hat{b}^*)$ and $\tilde{x}^* = (\tilde{w}^*, \tilde{b}^*)$ where $\tilde{b}^* > 0$. Then, $\hat{x}^*$ and $\tilde{x}^*$ are fixed points of Φ and $\hat{w}^* > \tilde{w}^* > \tilde{b}^* > \hat{b}^*$.

Proof. An induction argument using the properties of Φ gives $\hat{w}_n > \tilde{w}_n > \tilde{b}_n > \hat{b}_n > 0$ for all $n > 0$. Further, $\beta_n := \hat{b}_n/\tilde{b}_n$ satisfies $0 < \beta_{n+1} = \beta_n \psi(\hat{x}_n)/\psi(\tilde{x}_n) < \beta_n$ for $n \geq 0$. Thus, $\beta_\infty := \lim_{n\to\infty} \beta_n$ exists and $0 \leq \beta_\infty < 1$ implies $\hat{b}^* = \beta_\infty \tilde{b}^* < \tilde{b}^*$. We claim that $\tilde{x}^* \in \mathbb{X}$ which necessarily implies $\hat{x}^* \in \mathbb{X}$. Suppose $\tilde{w}^* = \tilde{b}^*$. By the boundary behavior of ψ, $\lim_{n\to\infty} \psi(\tilde{x}_n) = \infty$ which, since $\tilde{b}_n$ is bounded away from zero and $\tilde{w}_n$ from above, would imply $\tilde{b}_n > \tilde{w}_n$ for some n sufficiently large, a contradiction. Conclude that $\tilde{x}^* \in \mathbb{X}_+$. Continuity of Φ then implies $\lim_{n\to\infty} \tilde{x}_{n+1} = \lim_{n\to\infty} \Phi(\tilde{x}_n) = \tilde{x}^* = \Phi(\tilde{x}^*)$, i.e, $\tilde{x}^*$ is a fixed point of Φ. The argument for $\hat{x}^*$ is analogous. Finally, $\hat{w}^* > \tilde{w}^*$ by monotonicity of Φ. ■

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